Valuing Multiple Natural Capital Stocks Under Correlated Volatility

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Abstract

Bioeconomic models can be used to value single and multiple coupled natural capital stocks as assets under real-world management conditions. In this paper we extend prior work to consider the valuation of assets linked through deterministic relationships (i.e. biophysical coupling or shared management) to assets with stochastic dynamics including when there are multiple stock with correlated stochastic processes. We derive asset prices for natural capital stocks governed by correlated diffusions and show how function approximation techniques can be used to approximate these shadow prices across the domain of capital stocks. Using single- and multi-species examples, we demonstrate the combined role of biophysical dynamics, the management feedback rule, and the properties of the valuation function for benefits flows in influencing the salience of risk in the pricing of natural assets. Finally, we examine how the interplay between the deterministic links between capital stocks (i.e. through ecological interactions) and their covariance can enhance or dampen substitutability/complementarity relationships that are at the heart of the sustainable management dilemma.

Keywords: Natural Capital; Real Assets; Sustainability; Risk
JEL Codes: G12, G32, Q01, Q56, Q57

1 Introduction

What is the value to society of a change in a particular natural capital stock or portfolio of capital stocks? The answer to this question is important for two reasons. First, it provides the relevant quantity to use as the opportunity cost of the consumption of natural capital (i.e. clearing of an acre of wild lands or harvesting a ton of fish) for the purposes of social benefit-cost analysis. Second, the valuation of capital assets (especially natural capital) is critical to assessing the sustainability of the planet, its nation states, and bounded social-ecological systems. Maintenance of society's "productive capacity" - its ability to ensure non-declining intertemporal social welfare (i.e. the net present value of welfare flows) requires maintaining the overall portfolio value of society's assets (Dasgupta, 2001; Dasgupta and Mäler, 2000). This "inclusive wealth" is often presented as a linear index of the quantities of specific capital stocks, where the weights on these quantities are the monetized "accounting prices" or "shadow prices" for each capital stock. Yet, the prices themselves are functions of stock quantities. Therefore the inclusive wealth index is not truly linear in stocks and can capture substitution opportunities (Yun et al., 2017). These prices are forward-looking in the sense that they represent all the benefits and costs of an additional investment in a particular capital stock. They are also inherently grounded in a particular scenario (i.e. the "economic program") for how the capital stock (and any interrelated stocks) are managed today and will be managed into the indefinite future. Dasgupta (2001) has argued forcefully that in order for inclusive wealth to provide operational input to policy makers about the sustainability of their economies, shadow prices should avoid scenarios of optimality and efficiency that dominate the literature and instead be grounded in realistic "economic programs" that reflect existing imperfect institutions and policies.

Wealth-based measures of sustainability have gained substantial credibility beyond economists (e.g., Matson, Clark, and Andersson (2016)) and are regularly employed by the United Nations Environment Programme and the World Bank for the sustainability assessment of nation states (UNU-IHDP and UNEP, 2014). They have also been used to manage the sustainability of bounded systems such as cities (Dovern, Quaas, and Rickels, 2014), hydrological catchments (Pearson et al., 2013) and as an indicator of sustainable management for ecosystems (Yun et al., 2017). Nonetheless, the applicability of inclusive wealth has been hampered by the frequent lack of reliable values for the shadow prices of natural capital. Aside from a small number of exhaustible minerals and some forests, most forms of natural capital lack asset markets from which to obtain prices. Furthermore, many natural capital stocks provide service flows that are non-excludable and/or non-rivalrous and are managed in demonstrably inefficient, 'kakatopic' ways - forestalling the use of shadow prices from optimized bioeconomic models as a realistic guide for sustainability assessment. The result for wealth-based assessments is that many forms of natural capital have been excluded from accounts altogether - implicitly receiving shadow prices of zero.

In an attempt to address this 'Achilles' heel' of sustainability (Smulders, 2012), Fenichel and Abbott (2014) provided a rigorous derivation of the shadow price of natural capital under general, non-optimized forms of management, and link their derivation to foundation economic capital theory, a la Jorgenson. They showed how bioeconomic models combining the valuation of benefit flows from natural capital (i.e. ecosystem services), stock dynamics of natural capital, and feedbacks between capital stocks and human behavior (as shaped by institutions) can be combined with function approximation approaches to implement this valuation across the domain of natural capital stocks (Fenichel et al., 2016).[1]

This approach has subsequently been expanded to allow for the valuation of a portfolio of capital stocks whose dynamics may be interlinked through physical or biological processes or via human behavior (Yun et al., 2017). These methods have been used to value a range of natural capital stocks, from fish in single-species fisheries (Fenichel and Abbott, 2014), groundwater (Fenichel et al., 2016), coastal habitat (Bond, 2017), and an assemblage of interacting fish stocks (Yun et al., 2017).

Despite this significant progress, the theory and practice of natural capital valuation remain incomplete in that the shadow price formulae and approximation approaches fail to reflect the role of uncertainty in the evolution of natural capital stocks. Just as the volatility associated with the returns on a marketed asset is reflected in its share price, so should the shadow prices of natural capital reflect the uncertainties associated with its physical 'capital gains'. Model-based shadow prices are inherently dependent on the underlying specification of the model of natural capital dynamics. However, there is often significant uncertainty with

[1]See Fenichel, Abbott, and Yun (2018) for a detailed development of natural capital pricing.

regard to these models. Our understanding of many natural processes is at best incomplete, with the result that the actual evolution of natural capital could deviate significantly from any specific model. The valuation of such an inherently risky asset may differ significantly from one where the capital dynamics are deterministic and known with certainty. Furthermore, many natural capital stocks in a given system are differentially vulnerable to a wide array of systemic and idiosyncratic shocks, with the result that changes in their stocks may be correlated - even in the absence of fundamental interactions in their dynamics. Since sustainability requires managing the wealth contained in a *portfolio* of capital stocks, it may be important to sustainable management to understand how the properties of the covariance structure of stocks in the 'ecosystem fund', influences the overall value of the portfolio and how this correlated volatility interacts with the mechanistic interactions between capital stocks and the portfolio balancing decisions embodied in management policies.

In this paper, we extend the work of Fenichel and Abbott (2014) and (Yun et al., 2017) to derive the shadow values for single stock dynamics subject to a stochastic process and for multiple physically-interacting capital stocks when their dynamics are influenced by stochastic and correlated diffusion processes. We show how these shadow prices can be approximated, given a full bioeconomic model and specification of the diffusion process, through an extension of the functional approximation technique employed in (Yun et al., 2017). We first revisit the model used in Fenichel and Abbott (2014), but assume the stock dynamics are stochastic. Then, we demonstrate our approach using a simple model of two ecologically interacting species. Drawing upon modern portfolio theory, we emphasize how the stochastic dynamics of natural assets can interact in different ways with the deterministic, structural relations between these assets to influence the overall sustainability of the managed system. We show how the portfolio re-balancing conditions embodied in the 'economic program', a feedback rule denoting how stocks are depleted in response to the current state of the system, can be valued in expectation using the value functions and inclusive wealth measures that spring from our approach. The analyses provide two important insights. First, the influence of stochastic dynamics in single stock systems may be relatively small, building confidence in simpler deterministic analyses. Second, the results provide new insights into the insurance role of ecosystems.

2 Derivation of shadow pricing formula

2.1 The single asset case

Let $s(t)$ represent the known stock of a scalar capital stock at time t.[2] Suppose the dynamics of s are represented by a diffusion (also known as an Ito) process and stationary infinitesimal parameters $\mu(s, x(s))$ and $\sigma(s)$. The diffusion process is written as follows:

$$ds(t) = \mu(s(t), x(s(t)))\, dt + \sigma(s(t))dZ(t) \tag{1}$$

where $dZ(t)$ is an increment of a Wiener process (Stokey, 2009). Notice that the drift of the diffusion $\mu(s, x(s))$ is specified as a function of the current capital stock and as a function of the feedback control rule of the economic program, $x(s)$. There is no uncertainty in the behavioral feedbacks, although future work may consider extending the results to this case. This implies that stochasticity comes through the ecological production process. Once the substitution for the economic program has been made, the drift is an explicit function of only s.

Define the intertemporal welfare function, evaluated along the economic program and along the stochastic capital trajectory given by (1), as

$$V(s(t)) = \mathbb{E}_t \left[\int_t^\infty e^{-\delta(\tau - t)} W(s(\tau), x(s(\tau)))\, d\tau \right] \tag{2}$$

The marginal value of an investment in the capital stock in expectation is defined as $p(s) \equiv$

[2] t is suppressed when doing so does not cause confusion.

V_s. To derive the properties of $p(s)$, start by differentiating (2) with respect to t.

$$\frac{dV}{dt} = \mathbb{E}_t\left[\delta \int_t^{\infty} e^{-\delta(\tau - t)} W(\cdot)\, d\tau - W(s(t), x(s(t)))\right] = \delta V - W(s(t), x(s(t))) \qquad (3)$$

The first equality in (3) assumes that the derivative can be carried through the expectation operator, which is ensured by the stationarity of the infinitesimal parameters of (1). The second equality holds because the state of the system is known at $\tau = t$.

We know that $\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt}$. By Ito's Lemma

$$dV = \left[\mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss}\right] dt + \sigma(s) V_s dZ$$

Taking the expected value, and employing the property that all stochastic integrals are identically zero (Stokey, 2009):

$$\mathbb{E}_t[dV] = \left[\mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss}\right] dt$$

so that

$$\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt} = \mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss} \qquad (4)$$

Setting (3) equal to (4) we obtain the stochastic Hamilton-Jacobi-Bellman (HJB) equation:

$$\delta V(s) = W(s(t), x(s(t))) + \mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss} \qquad (5)$$

If we substitute $p(s) \equiv V_S$ into the HJB equation yielding:

$$\delta V(s) = W(s(t), x(s(t))) + p(s)\mu(s) + \frac{1}{2}\sigma^2(s) p_s(s) \qquad (6)$$

The first two terms on the RHS are the traditional deterministic current-value Hamiltonian. The third term captures the effect of risk even if the deterministic rate of change in the capital stock $\mu(s) = 0$. The risk effect captures the effect of Jensen's inequality via the curvature of the value function. If the shadow price function is downward sloping then $p_s < 0$ so that risk has a negative effect on the value function. Suppressing functional dependency on s, and differentiate (6) with respect to s yields:

$$\delta p = W_s + \mu_s p + \mu p_s + \sigma\sigma_s p_s + \frac{1}{2}\sigma^2 p_{ss}$$

Isolating p on the left-hand side we obtain the asset pricing equation:

$$p(s) = \frac{W_s + [\mu(s) + \sigma(s)\sigma_s(s)]p_s + \frac{1}{2}\sigma^2(s)p_{ss}}{\delta - \mu_s(s)} \tag{7}$$

In the case where the variance of the noise in (1) does not depend on s then (2.1) reduces to:

$$p(s) = \frac{W_s + \mu(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}}{\delta - \mu_s(s)}$$

and if capital dynamics are deterministic then this further reduces to

$$p(s) = \frac{W_s + \mu(s)p_s}{\delta - \mu_s(s)}$$

which is the same as in (Fenichel and Abbott, 2014) who linked this equation to Jorgenson (1963).

The general asset pricing equation equation (7) contains two additional numerator terms relative to Fenichel and Abbott's deterministic derivation. The first term enters in a way that is symmetric to capital gains in a deterministic system and depends on the extent of "risk aversion" embodied in the curvature of of the intertemporal welfare function (since $p_s \equiv V_{ss}$) and the extent to which the standard deviation of the diffusion is elastic with respect to s. If increasing investment in s increases the size of shock, and if the shadow price function is decreasing in the stock (analogous to risk aversion), then this results in a "capital loss." This term only matters if the variance depends on the capital stock, as in the case of geometric Brownian motion. Importantly, curvature of the intertemporal welfare function, which is defined over the domain of capital *stocks*, need not result from underlying curvature of the "social utility" or real income function for welfare flows $W(\cdot)$. Indeed, the nature of risk preferences over flows embodied in W (including risk neutrality) may have no direct mapping to the curvature of $V(s)$. Curvature of the intertemporal welfare function can be inherited from the underlying biophysical dynamics in (1) or from the economic program $x(s)$ - suggesting that the risk premia embodied in the numerator of (7) are endogenous to policy and may reflect actual existing levels of self-insurance and self-protection. This first term pertains to how a marginal investment in the capital stock increases risk, holding the curvature of the intertemporal welfare function constant.

The second additional term in (7) is always present with stochastic dynamics and depends on the properties of the third derivative of the value function. There will be a premium if there is a positive third derivative (convex price function), while a negative third derivative (concave price function) yields a discount. If the value function is quadratic (i.e. zero derivatives above the second derivative), then this term is zero. Both additional terms in the numerator of (7) originate from differentiating $\frac{1}{2}\sigma^2(s)p_s$ term in (6). This second can be interpreted as the affect of a marginal increase in the capital stock on risk aversion, holding risk constant. If risk aversion is increased by the investment ($V_{sss} = p_{ss} < 0$) then the shadow price is decreased. In other words, the pricing of risk into the capital asset depends on how an investment affects the sensitivity to risk, given the biophysical dynamics and economic program in place, in addition to how the marginal investment affects the risk itself. This can be thought of as an "insurance effect."

2.2 The multi-stock case

Let $\boldsymbol{s}(t) \in \mathbb{R}^S$ and $\boldsymbol{x}(\boldsymbol{s}(t)) : \mathbb{R}^S \to \mathbb{R}^X$ and extend the diffusion in (1) to S distinct Ito processes

$$ds^i = \mu^i\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)dt + \sigma^i(\boldsymbol{s})dZ^i(t) \quad \text{for } i = 1, \ldots, S \tag{8}$$

The $dZ^i(t)$ can be correlated with a $S \times S$ correlation matrix $\boldsymbol{\rho}$ such that the covariance of the stochastic components of capital stocks i and j, which may differ from their observed covariance in-sample due to the presence of deterministic relations between the stocks in (8), is $\mathbb{E}_t\left[\sigma^i(\boldsymbol{s})dZ^i(t)\sigma^j(\boldsymbol{s})dZ^j(t)\right] = \sigma^i(\boldsymbol{s})\sigma^j(\boldsymbol{s})\mathbb{E}_t\left[dZ^i(t)dZ^j(t)\right] = \sigma^i(\boldsymbol{s})\sigma^j(\boldsymbol{s})\rho^{ij}dt$. Notice that if $i = j$ this boils down to $\sigma^i(\boldsymbol{s})^2 dt$.

While the decomposition of the noise into a correlation matrix and standard deviations is intuitive and useful for model parameterization intuition, we work directly with the covariance matrix to conserve on notation. Therefore, let $\Omega(\boldsymbol{s})$ be a $S \times S$ covariance matrix of the noise terms such that $\text{Cov}(ds^i, ds^j) = \Omega^{ij}(\boldsymbol{s})\, dt$. A Cholesky decomposition of the covariance matrix yields $\Omega(\boldsymbol{s}) = \omega(\boldsymbol{s})\omega(\boldsymbol{s})'$.[3]

Redefine the instantaneous return functions and intertemporal welfare functions in the multi-stock case as $W\left(\boldsymbol{s}\left(t\right), \boldsymbol{x}\left(\boldsymbol{s}\left(t\right)\right)\right)$ and $V(\boldsymbol{s}(t))$. Once again, we know that $\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt}$. Applying Ito's Lemma (Dixit and Pindyck, 1994) yields:

$$dV(\boldsymbol{s}) = \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] dt + \sum_{j=1}^{S} \sigma^j(\boldsymbol{s}) V_{s^j} dZ^j$$

Finding the expected value and dividing through by dt:

$$\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt} = \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] \tag{9}$$

Setting (9) equal to the multidimensional generalization of (3) yields the HJB equation.

$$\delta V(\boldsymbol{s}) = W\left(\boldsymbol{s}\left(t\right), \boldsymbol{x}\left(\boldsymbol{s}\left(t\right)\right)\right) + \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] \tag{10}$$

Partial differentiation of (10) yields the following expression for the shadow price of s^i

[3] This approach generalizes (8) slightly by technically allowing for the *correlation* matrix - not just the standard deviations - to vary in the stock vector.

$$p^i(\boldsymbol{s}) = \frac{W_{s^i} + \left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^S \frac{\partial p^j}{\partial s^i}\mu^j\right) + \sum_{j\neq i}^S p^j \mu^j_{s^i} + \frac{1}{2}\sum_j^S \sum_k^S \left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)}{\delta - \mu^i_{s^i}}$$

Factoring the final numerator term yields the final asset pricing equation.

$$p^i(\boldsymbol{s}) = \Bigg[W_{s^i} + \left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^S \frac{\partial p^j}{\partial s^i}\mu^j\right) + \sum_{j\neq i}^S p^j \mu^j_{s^i} + \frac{1}{2}\sum_{j=1}^S \left(\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j} + \sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}\right)$$
$$+\frac{1}{2}\sum_{j=1}^S \sum_{k\neq j}^S \left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)\Bigg] \Bigg/ \left(\delta - \mu^i_{s^i}\right) \quad (11)$$

The first numerator term in (11) has the same interpretation as in the single-asset case. The next two terms in the numerator are present in the deterministic multi-asset case (Yun et al., 2017) and are forms of "capital gains." The second numerator term $\left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^S \frac{\partial p^j}{\partial s^i}\mu^j\right)$ reflects the effects of investment in s^i on the shadow price of stock i due to its prices of all assets in the portfolio (i.e. "price effects"). The third numerator term $\sum_{j\neq i}^S p^j \mu^j_{s^i}$ captures the deterministic effects of investment in stock i on the physical growth rates of all other stocks ("cross-stock effects"), which can stem from system ecology or production interactions within the economic program.

The additional numerator terms in (11) are only present in the stochastic case. The third term $\frac{1}{2}\sum_{j=1}^S \left(\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j} + \sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}\right)$ operates solely through the individual variances of each asset and captures the "risk sensitivity" effect of an investment in asset i on the variance of each asset, $\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j}$. Importantly, this term accounts for the fact that investments in i generally affects the *sensitivity* to risk for all S assets, $\sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}$, even if the variance for these other assets remains unchanged by the investment, the "insurance effect." Together, these terms mirror the numerator terms, $\sigma(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}$, in (7).

The final term in the numerator of (11), $\frac{1}{2}\sum_{j=1}^S \sum_{k\neq j}^S \left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)$, reflect the risk-related effects of investing in asset i that are mediated through the *covariances* of assets in the portfolio. This term is zero in the case that natural capital stocks are uncorrelated regardless of the vector of capital stocks. $\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k}$ is the effect of an investment in i on the covariances between other assets j, k as valued through the first cross-partial between these assets (i.e. the 2nd cross-partial of the intertemporal welfare function). If the covariances between asset stocks are invariant to capital stocks then this term is zero. $\Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}$ reflects the fact that investing in i may itself affect the curvature of the intertemporal welfare function in the direction of k and i (i.e. $\frac{\partial^2 p^j}{\partial s^k \partial s^i} = \frac{\partial}{\partial s^i}V_{s^j s^k}$). If the effect of increasing asset i is to increase the concavity in the direction of increases in j and k ($\frac{\partial^2 p^j}{\partial s^k \partial s^i} < 0$) then the existence of positive correlation between the latter two assets results in a compensating reduction in the asset price. This creates addition "insurance" opportunities.

Some insight on the numerator terms involving covariances can be gleaned by realizing that the covariance between innovations in s^j (the residual of changes in s^j after the deterministic drift $\mu^j(\boldsymbol{s}, \boldsymbol{x}(\boldsymbol{s}))$ is differenced away) and innovations in s^k can be viewed as their rescaled relationship in expectation. Specifically, if the conditional expectation of s^j and s^k is linear[4] $\mathbb{E}[ds^j | ds^k] = \beta ds^k$, then it is well known that $\beta = \frac{\Omega^{jk}}{\sigma^{2k}}$. In other words, the covariance terms in (11) reflect the expected marginal effect of ds^k on ds^j such that the risk terms in the multivariate asset case account for systematic (linear) cross-effects between perturbations in stocks in a way that is analogous to how the previous cross-terms in the numerator account for capital gains through deterministic relationships via price and cross-stock effects.

Finally, it is noteworthy that the effects of stochasticity disappear from (11) when two conditions hold: 1) when all second moments are constant regardless of the stock levels, and 2) the intertemporal welfare function, V, is quadratic such that investments have no effect on its curvature. However, since the intertemporal welfare function inherits the properties of the instantaneous benefits function, the economic program, and biophysical dynamics in a complex manner, the latter property is difficult to verify *ex ante*.

[4]Linearity of the conditional mean follows directly from the joint normality assumption for Ito processes.

3 Numerical approximation

Given a complete deterministic bioeconomic model of a social-ecological system it is possible, at least in principle, to obtain approximate shadow values for a given stock at a given initial state vector by perturbing the desired natural capital stock and calculating the change in the net present value of benefits flows over the indefinite future. While straightforward, this approach is computationally intensive and cumbersome for forecasting or backcasting the wealth dynamics of a system and may be inappropriate in stochastic settings. Fenichel, Abbott, and Yun (2018) and Yun et al. (2017) describe how the HJB equation can be combined with functional approximation approaches frequently used in numerical dynamic programming to approximate the entire shadow price *function* over a closed domain of capital stocks. For the deterministic, multi-asset case they advocate approximating $V(\boldsymbol{s}(t))$ using the HJB equation (analogous to (10)), replacing $V(\boldsymbol{s}(t))$ on the LHS of the equation with a weighted sum of the tensor product of Chebychev basis functions in the stock vector $\boldsymbol{s}(t)$ and replacing the partial derivatives of the value function on the RHS with the partial derivatives of this approximation. The coefficients that determine the weightings on the basis functions can be solved analytically and are chosen (in a system with as many approximation points as coefficients) to make the LHS and RHS of the approximated HJB equation hold with equality.[5]

This value function approximation technique can be adapted with relatively minor changes to the stochastic diffusion case. First, define the bounded approximation interval for each state variable. Then choose M evaluation points within this interval for each of the S capital stocks and then calculate $W\left(\boldsymbol{s}\left(t\right), x\left(\boldsymbol{s}\left(t\right)\right)\right)$, $\mu\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)$ and $\Omega(\boldsymbol{s})$ at each point.[6] The univariate node coordinates are then permuted to yield M^S grid points. We define ϕ^i as

[5]In some cases it may be desirable to utilize more approximation nodes than the number of coefficients - an overdetermined system. In this case, the coefficients can be chosen to minimize the sum of squared deviations between the LHS and RHS of the approximation. The analytical expression for this solution is analogous to ordinary least squares (Fenichel, Abbott, and Yun, 2018).

[6]In many cases the evaluation nodes are found by finding the M roots of a unidimensional Chebyshev polynomial on the bounded approximation range for each state variable.

the $M \times (q^i + 1)$ basis matrix of q^ith degree for state variable i. This is a matrix of $q^i + 1$ basis functions - Chebyshev polynomials of ascending degree in our case - evaluated at the M evaluation points. To approximate over the bounded domain in $\mathbb{R}^S$ we find the tensor product across all dimensions (i.e. allow for full interactions across the univariate basis functions) to form an $M^S \times \prod_{i=1}^{S} (q^i + 1)$ basis matrix: $\boldsymbol{\Phi}(\boldsymbol{S}) = \phi^N \otimes \phi^{N-1} \otimes \ldots \otimes \phi^1$ where $\boldsymbol{S}$ is the $M^S \times S$ matrix of evaluation points (i.e. all grid nodes of M evaluation points for all S state variables). We can now define our approximation to the intertemporal welfare function $V(\boldsymbol{S}^m) \approx \boldsymbol{\Phi}^m(\boldsymbol{S})\boldsymbol{\beta}$ where m indexes the M^S distinct capital stock vectors (i.e. the individual evaluation points in the S-dimensional grid) and $\boldsymbol{S}^m$ is the mth row of $\boldsymbol{S}$. $\boldsymbol{\Phi}^m(\boldsymbol{S})$ is the mth row of $\boldsymbol{\Phi}(\boldsymbol{S})$, and $\boldsymbol{\beta}$ is a $\prod_{i=1}^{S} (q^i + 1) \times 1$ vector of unknown approximation coefficients. Using the fact that $\frac{\partial V(\boldsymbol{S}^m)}{\partial s^i} \approx \left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^i}\right)\boldsymbol{\beta}$ and $\frac{\partial^2 V(\boldsymbol{S}^m)}{\partial s^i \partial s^j} \approx \left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^i \partial s^j}\right)\boldsymbol{\beta}$ we can replace the HJB equation in (10) with the following approximation:

$$\begin{aligned}\delta\boldsymbol{\Phi}^m(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S}^m) + \Bigg[&\sum_{j=1}^{S} diag(\mu^j(\boldsymbol{S}^m))\left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j}\right)\boldsymbol{\beta} \\ &+\frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} diag(\Omega^{jk}(\boldsymbol{S}^m))\left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j \partial s^k}\right)\boldsymbol{\beta}\Bigg]\end{aligned} \tag{12}$$

Collecting terms involving β yields:

$$\left[\delta\boldsymbol{\Phi}^m(\boldsymbol{S}) - \sum_{j=1}^{S} diag(\mu^j(\boldsymbol{S}^m))\left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j}\right) - \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} diag(\Omega^{jk}(\boldsymbol{S}^m))\left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j \partial s^k}\right)\right]\boldsymbol{\beta}$$

$$= \boldsymbol{\Psi}^m(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S}^m)$$

Stacking these M^S vector equations results in the equation $\boldsymbol{\Psi}(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S})$. If $M^S = \prod_{i=1}^{S} (q^i + 1)$ (i.e. the number of approximation points equals the number of unknown approximation coefficients) then the approximation coefficients can be calculated in a straightfoward way through matrix inversion. Alternatively, if $M^S > \prod_{i=1}^{S} (q^i + 1)$ then the $\boldsymbol{\beta}$ can be found using least squares.

$$\boldsymbol{\beta} = \left(\boldsymbol{\Psi}(\boldsymbol{S})'\boldsymbol{\Psi}(\boldsymbol{S})\right)^{-1} \boldsymbol{\Psi}(\boldsymbol{S})'W(\boldsymbol{S}) \tag{13}$$

After obtaining the approximation $\boldsymbol{\Phi}(\boldsymbol{S})$ it is straightforward to find the shadow values of any given capital stock by taking its partial derivative.

Fenichel, Abbott, and Yun (2018) discuss the importance of determining the domain of approximation. They show that in multi-dimensional systems the system dynamics to can lead outside the approximation domain, which hinders the ability to recover shadow prices. They argue that it is important to make sure the approximation domain is sufficient to include dynamic from any stock size for which a shadow price is desired. In the single stock deterministic case this is never an issue so long as the system has attractors that interior to the approximation domain. However, this property does not extend to stochastic dynamics. This is because a shock at the edge of the approximation domain could lead the system outside the approximation domain for a non-trivial period of time. Therefore, extra attention is needed to enlarge the approximation domain when system dynamics are stochastic.

4 Single stock example

The single-species example focuses on understanding the circumstances in which incorporating stochastic capital dynamics is important for assessing the value of natural capital and wealth-based sustainability. In particular we examine the extent to which the biophysical stochastic dynamics coupled with the economic program may yield substantially different shadow prices from the deterministic case and the mechanisms for this divergence through either the 'endogenous risk' or 'endogenous risk aversion' effects discussed above.

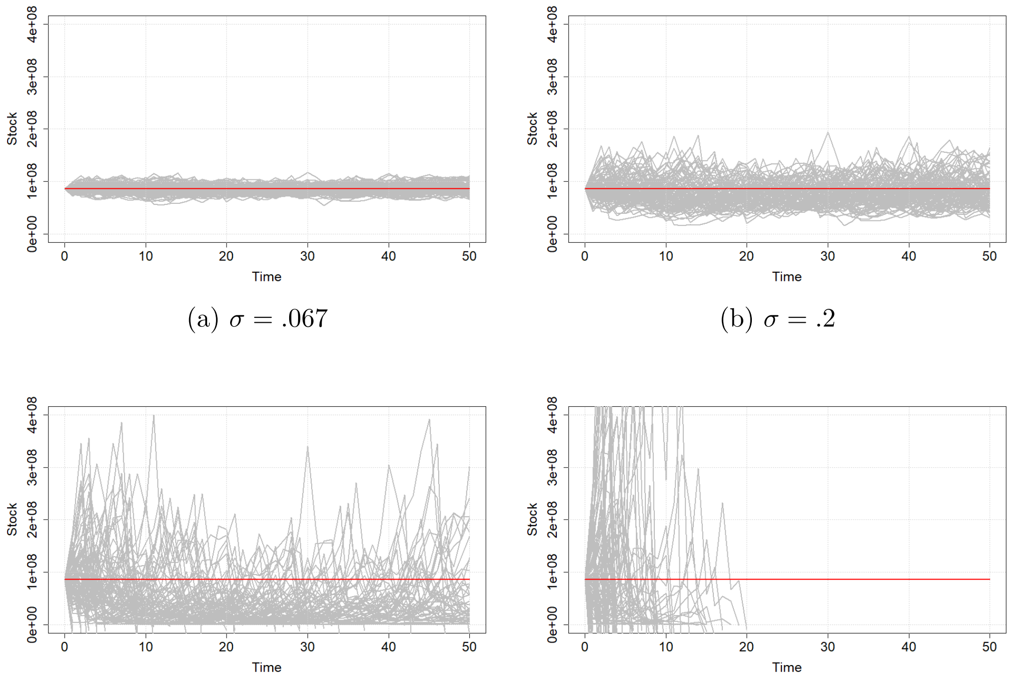
We extend the deterministic model for Gulf of Mexico reef fish presented in Fenichel and Abbott (2014) and originally developed by Zhang and Smith (2011) and Zhang (2011) to the stochastic case. We augment the logistic stock dynamics with an additive geometric Brownian motion (GBM) noise term. Geometric Brownian motion allows the standard deviation of stock perturbations to scale linearly with the stock level and is consistent with the assumptions of log-normal disturbances frequently used in population dynamic modeling and fisheries stock assessment. Utilizing historic assessed biomass data from the fishery we calibrate $\sigma = 0.067$; therefore the standard deviation from the deterministic drift given by the logistic growth equation with harvest is approximately 6.7% of the stock level. The stock dynamics are

$$ds = \left(0.3847s(t)\left(1 - \frac{s(t)}{3.59 \times 10^8}\right) - h\left(x\left(s\left(t\right)\right), s\left(t\right)\right)\right)dt + 0.067s(t)dZ(t) \tag{14}$$

The economic program, the relationship linking stock status (in lbs.) and effort (in crew-days) in the fishery, is provided by a power rule, $x(s) = ys^{\gamma}$, where $\gamma = 0.7882$ and $y = 0.157$. We assume that the valuation of income flows in the fishery is directly expressed in terms of monetary profits, with price-taking firms and costs that are linear in effort: $W = mh - cx$, with m = \$2.70/lb., c = \$153/crew-day. The production function for harvests is of a generalized Schaefer form $h = qsx\left(s\right)^{\alpha}$, with $q = 3.17 \times 10^{-4}$ and $\alpha = 0.544$. Note that while both harvests and costs are linear in the fish stock for a fixed allocation of effort x, $W(s)$ is actually a strictly *convex* function of the stock once the endogenous feedback from the stock level to harvest behavior $x(s)$ is incorporated.

4.1 The effects of risk, σ

Figure 1 provides stochastic simulations of stock paths originating from the steady state biomass and harvest under increasing levels of stochasticity. The level of noise introduced by stochasticity in the base case (Fig. 1a) is already substantial and reminiscent of the noise in many ecological projections. The signal to noise ratio at the deterministic equilibrium stock is 3.2. Note that, while extinction is technically impossible in continuous time using



(a) $\sigma = .067$ (b) $\sigma = .2$

(c) $\sigma = .5$ (d) $\sigma = 1$

Figure 1: Stochastic simulations of stock dynamics over a range of values for σ. Note that the values for $\sigma = 1$ exceed the range of the graph on a number of runs.

the current economic program and geometric Brownian motion, our numerical simulations nevertheless show that the number of paths that crash to a numerically zero level increase dramatically with σ. Indeed, all paths reach numerical extinction within 20 periods when $\sigma = 1$. This suggests that levels of σ of 0.5 or 1 are likely inconsistent with the dynamics of most real-world species.

Figure 2 plots the approximated value function across the varying levels of σ in comparison to the deterministic case. As the HJB equation in Eq. (6) shows, the value function in the stochastic case includes an additional risk term that serves, in part, to shift the value function. In the current case, the value function is concave in s (i.e. the shadow price curve is downward-sloping) so that increasing σ in the GBM has the effect of reducing the expected net present value at any given stock level. This adjustment is very small for the empirically-justified level of stochasticity in our system ($\sigma = .067$) - suggesting that the

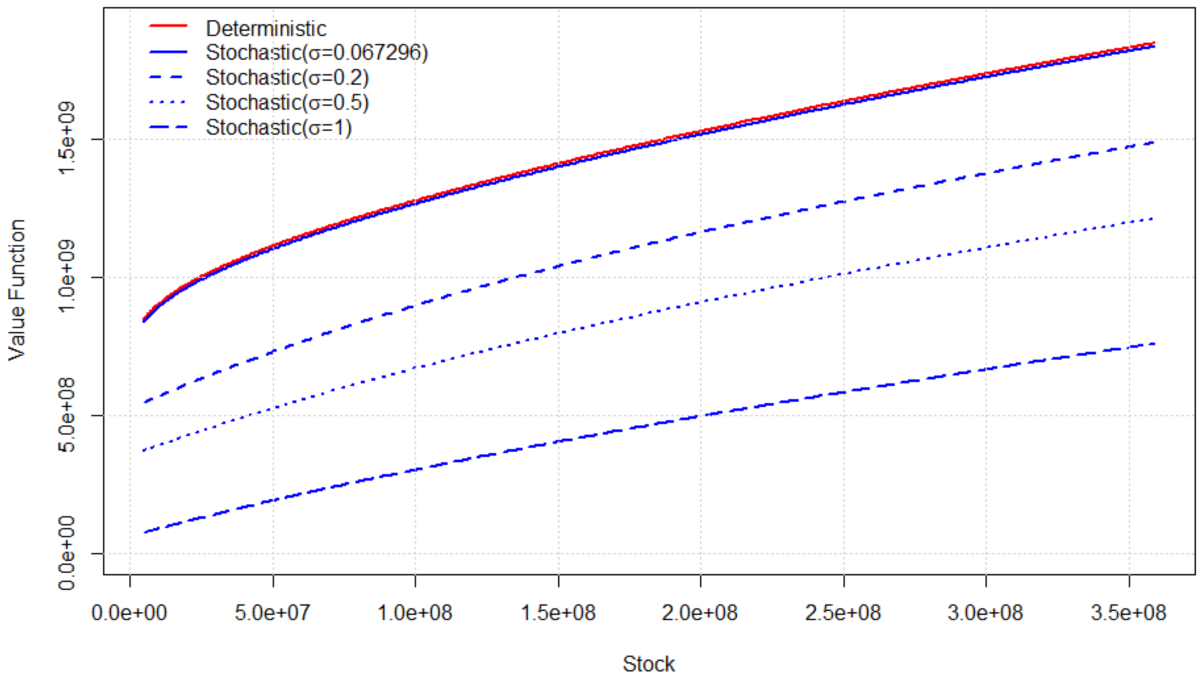


Figure 2: The value function for different values of σ

economic program is fairly robust to the level of stochasticity in the system by maintaining stock levels in a relatively insensitive range of the profit function. However, higher levels of risk lead to far less controllable systems, resulting in dramatic devaluations of the total 'ecosystem portfolio' by a factor of 1/2 to 2/3.

While the vertical shifts in the value function are the most obvious effect of varying σ there are other higher-order effects on the shape of the value function. The shadow prices (i.e. the derivative of the value function) are also changing, as reflected by the fact that the curves in Fig. 2 are not merely vertical displacements of each other. As shown in Eq. (7), these effects on the shadow price can occur either through endogenous risk or endogenous curvature of the value function. Figure 3 shows the shadow price functions, essentially the revealed 'demand functions' for natural capital, across the four risk levels. Interestingly, the price curve is effectively unchanged by the introduction of risk at the baseline level $\sigma = .067$. Thus, while risk devalues the stock in *total* (albeit very mildly) it has no appreciable effect on its *marginal* valuation.

This inelasticity of shadow prices to the level of risk changes at higher risk levels. Figure 3 demonstrates that as stochasticity in capital dynamics increases, the shadow price becomes more homogeneous across stock levels, reflecting a reduction in the curvature of the underlying value function. Indeed, the flattening of the shadow price curves with increased σ in Fig. 3 is evidenced by the increasing linearity of the value functions in Fig. 2. The homogenization of shadow prices across stock levels occurs because as σ increases the mapping between a given stock level and the subsequent trajectory of harvests and fishery profits becomes increasingly "fuzzy" and less bound to the deterministic trajectories of the deterministic case. In this case the shadow value of investing in an additional unit of natural capital at a current stock s_0 reflects the probability-weighted payoffs of all potential trajectories conditional on s_0. As σ increases the stock dynamics become more and more like a pure lottery, so that the conditioning on initial values becomes less and less important.

While the "homogenization effect" of risk primarily acts on the slope of the shadow price function, these changes are also accompanied by a general downward shift in the *level* of the shadow price curves - particularly for values of σ above 0.2. This "devaluation effect" suggests that an increase in stochasticity reduces the marginal value of stock investments at all levels. This occurs because the increasing role of noise in the system dynamics undermines the marginal value of holding capital. The possibility of either catastrophe or windfall are increasingly exogenous. The devaluation effect is most dramatic at low stock sizes since high levels of volatility, e.g., $\sigma = 1.0$ (Fig 3) lead to a high probability of near-extinction in the short to medium term whereas larger stocks are unlikely to be visited.

While we have demonstrated the potential for serious consequences of stochasticity for natural capital valuation, we have done so under arguably unrealistic assumptions. Indeed, the effects of risk on the shadow price in the previous example come almost entirely through the endogenous risk effect - where large values of σ in the GBM lead to outsized marginal effects of s on stock volatility. The third derivatives of the value function that figure in

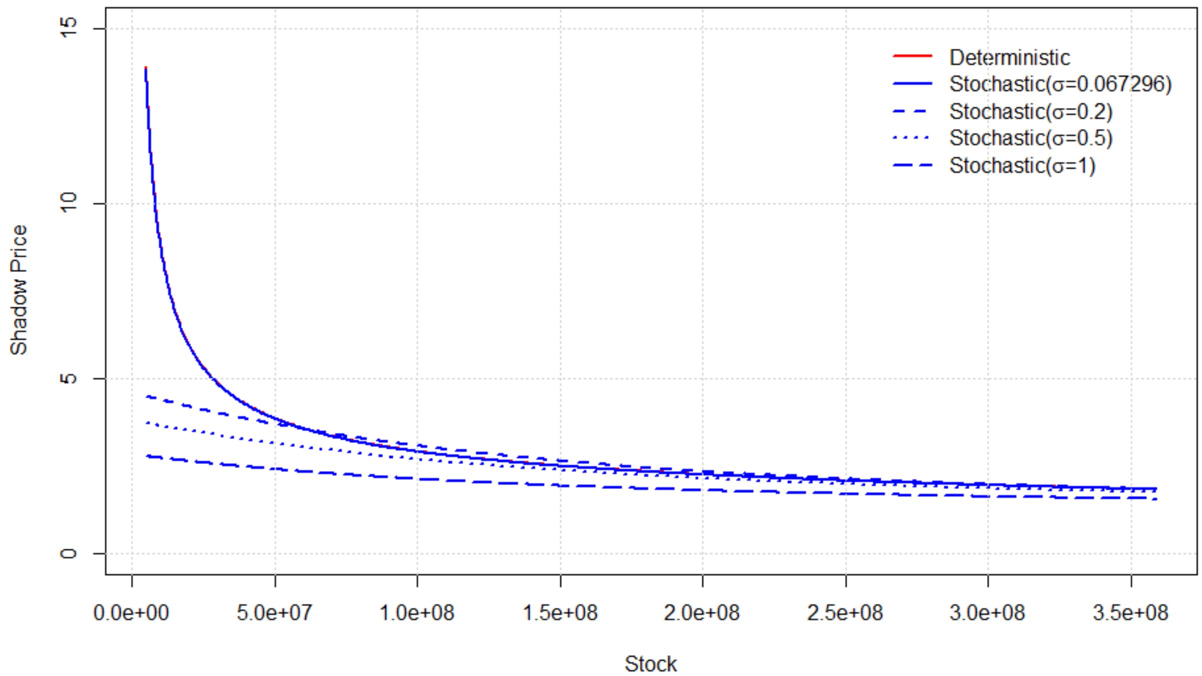


Figure 3: Shadow price for the Gulf of Mexico reef stock with increasing values of the volatility parameter σ.

the 'endogenous risk aversion' effect are negligible and become even more so as the value function becomes increasingly linearized at higher levels of risk. The negligible effects of risk on the value function at the empirically measured level $\sigma = .067$ suggests that there is precious little reason to incorporate stochasticity in capital dynamics when valuing natural capital in this particular system.

4.2 The effect of risk aversion in income flows

The previous section illustrates how the combination of biophysical dynamics, production functions, and feedbacks in the behavioral control rules between capital stocks and human behavior can result in a non-linear value function over stocks. This non-linearity, coupled with stock-dependent variability in natural capital stocks, can make risk relevant for asset pricing even when the underlying flows of income are valued in a risk-neutral manner. The

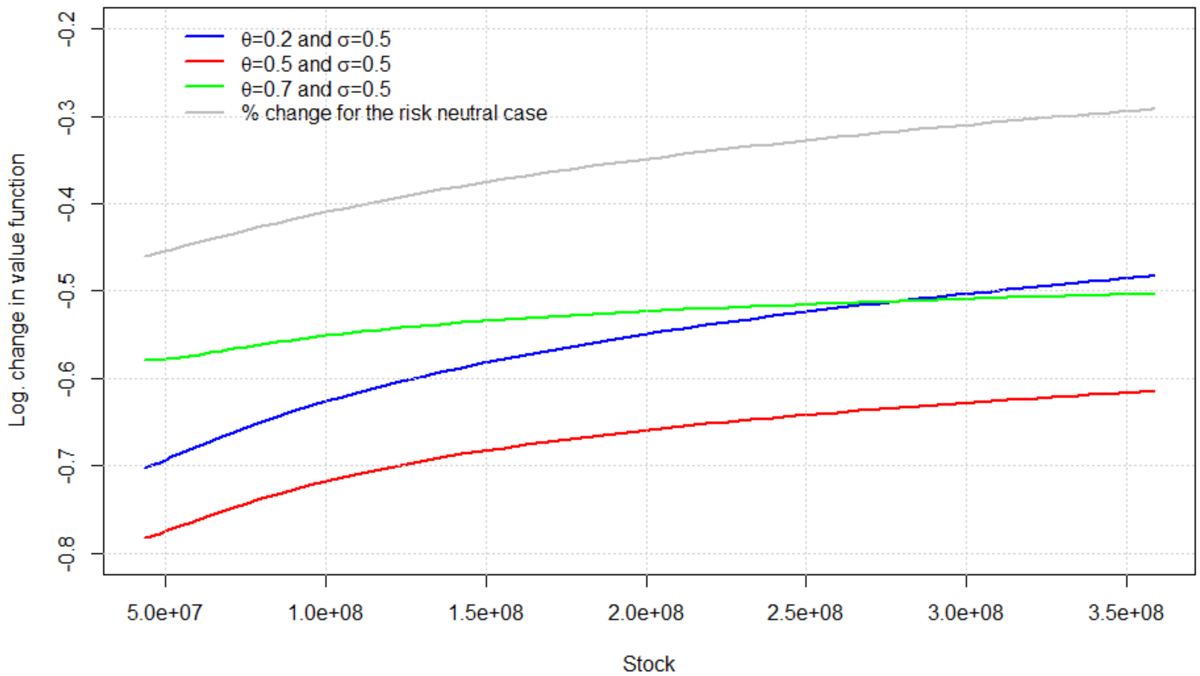


Figure 4: Logarithmic changes in value function for a perturbation of risk to $\sigma = 0.5$ relative to the baseline risk-free case for varying levels of risk aversion θ.

role of risk in the valuation of natural assets is therefore highly contingent on the totality of the features of the coupled social-ecological system.

Nevertheless, given this complexity, how does the explicit introduction of risk aversion over *flows* of income affect the role of risk in the pricing of natural capital? To investigate this question we replace $W(s)$ with its risk-adjusted form as a CRRA utility function in income flows.

$$W^{\text{risk}} = \frac{W(s)^{1-\theta} - 1}{1-\theta}$$

for $\theta \neq 1$. Note that larger positive values of θ imply greater aversion to variability in income flows.

Direct comparisons of value functions or shadow values across different levels of θ are not meaningful since their units are defined in terms of W and are therefore changing as θ changes. A more sensible comparison is to examine how the value function and shadow price

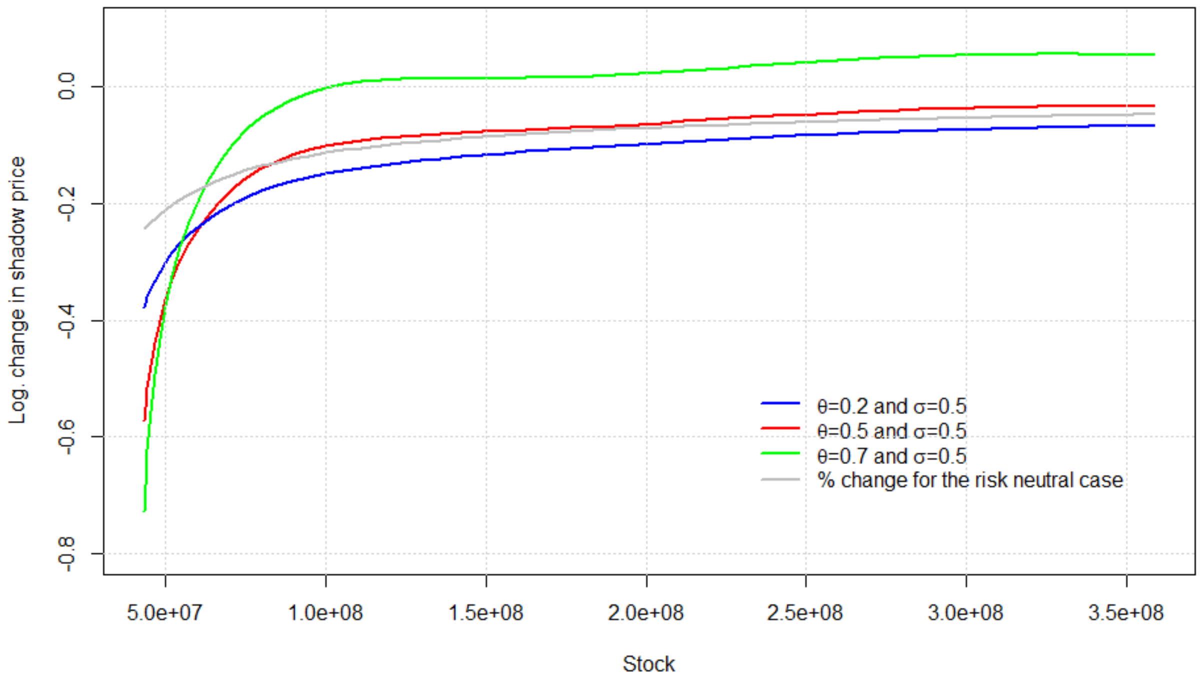


Figure 5: Logarithmic changes in shadow prices for a perturbation of risk to $\sigma = 0.5$ relative to the baseline risk-free case for varying levels of risk aversion θ.

functions change in *relative* terms in response to a perturbation in the level of risk for different levels of risk aversion. Figures 4 and 5 presents the change in the value function and shadow prices of an increase in risk to $\sigma = 0.5$ relative to a baseline deterministic ($\sigma = 0$) scenario. The most immediate finding from Fig. 5 is that there is not a uniform mapping between the level of risk aversion to income flows and the relative change in the value function from the introduction of risk. While the risk neutral case consistently has the smallest discount, the next smallest reduction is for the *most* risk averse case with $\theta = 0.7$ while the largest discount is consistently seen for an intermediate level of risk aversion ($\theta = 0.5$).

Turning attention to the relative changes in shadow prices (Fig. 5), the picture remains complex. For the most part shadow prices all decline with the introduction of risk. However, this is not the case for the highest examined level of risk aversion ($\theta = 0.7$), which actually shows *increases* in shadow prices for moderate to large stock sizes. The relative ranking of the risk discounts/premia across risk aversion levels are not consistent across stock sizes.

The exact cause of these complex findings remains a subject of research. Nevertheless, we believe that the explanation may relate to the fact that increases in stochasticity result in broader probabilistic mixing of dynamic trajectories of harvest, stocks and profits. This means that the value function, as the expected net present value of profits from these trajectories, probabilistically weights information about realized benefits flows W across the entire state space. Given that the value function under stochasticity conveys this global information at each 'local' value of the stock, the lack of crisp results is perhaps not surprising. At the very least, our results suggest that the mapping between static risk aversion to flows of income from capital stocks and the risk adjustment to value functions and shadow prices under non-optimized economic programs is not simple.

5 Conclusion

Uncertainty is a driving concern in natural resource economics and real-world resource management. Managers face an array of distinct risks, including measurement error and process error in terms of the future dynamics of natural stocks and human depletion of these stocks (i.e. uncertainty in the economic program). There is an enormous literature applying stochastic control theory to understanding the optimal management of resources (e.g., Sethi et al., 2005). There is also a growing literature applying modern portfolio theory to design optimal portfolios of harvested species or portfolios of spatial conservation across landscapes or seascapes according to the social planner's risk-return preferences (e.g., Ando and Mallory, 2012). However, the literature has devoted little or no attention to the question of how process error should be incorporated into the valuation of individual natural capital stocks or portfolios of these stocks under real world, non-optimized conditions. Addressing these challenges is critical for developing wealth-based metrics of sustainable development and sustainable resource management that treat natural resources as real assets that are linked both through their deterministic interactions and their stochastic properties.

We show how uncertainty over the future dynamics of natural resources can be rigorously incorporated into the shadow prices of assets at risk. While the forecast of resource trajectories can only be described in distributional terms, we show how this uncertainty can nevertheless be collapsed into a single shadow price at any given stock level. This finding parallels financial markets that yield a price for traded assets conditional on the information-contingent forecasts in the minds of traders - even as the flow of dividends from these assets is uncertain.

In the case of single assets, we find that risk enters into the valuation of natural capital in two ways. The first, an "endogenous risk" effect, reflects how capital investment affects its variability. This effect is valued through the curvature of the value function. The second, an "endogenous risk aversion" effect, reflects how these same investments affect the *valuation* of the risk by moving from regions of the value function with different degrees of curvature. Importantly, the valuation of risk depends critically on the second and third derivatives of the value function, which depends on the totality of the properties of the utility function valuing income flows from capital stocks, the shape of the growth functions of capital stocks, and the feedback rules between capital stocks and human behavior embodied in the economic program. In other words, the extent of "intertemporal risk aversion" - the curvature and change in curvature of the value function with respect to *stocks* of capital - is not "baked in" solely through the curvature of the utility function evaluating income flows (i.e. ecosystem services) from natural capital. Rather, it is a global property of the coupled human-natural system in question, including its management. Even in the case of a single natural asset, the feedback rule employed to respond to changes in the resource stock can affect the level of objective risk faced and the sensitivity of intertemporal welfare to that risk. In other words, resource management has a role to play in shaping the risk discount or premium. Indeed, this is logic behind the Shogren and Crocker's endogenous risk framework and broader literature on self-protection, self-insurance, and market based insurance.

This logic carries over to the multi-asset case, but in an even richer form. Investments in a given capital stock now have the potential to affect both the variances and the covariances of other capital stocks in the portfolio and the multi-dimensional curvature of the value function. These "portfolio effects" elevate the role of the economic program even further. The feedback rule between the vector of capital stocks and human actions on these stocks serves as a portfolio rebalancing rule that influences the overall value of the portfolio. The valuation approach we have outlined provide a metric for understanding how alternative portfolio management strategies influence the valuation of individual capital stocks and the sustainability of management itself - as reflected in forward simulations of inclusive wealth or the value function.

We have focused on analyzing stochasticity within the stock dynamics. Science often focuses on this form of uncertainty. However, there are other forms of uncertainty and stochasticity that may be equally salient. While we have focused on correlated diffusions in continuous time - which are inherently continuous in nature - there is also the possibility of resource dynamics experiencing discontinuous Poisson shocks or falling into an alternative basin of attraction. While these forms of risk fall outside the class of correlated diffusions considered here, we conjecture that they can nevertheless be handled in a relatively straightforward manner through extensions of analogues in the literature (Walker et al., 2010; Reed and Heras, 1992).

Other forms of stochasticity may be more difficult to conceptualize and measure. For example, the economic program could itself be uncertain or subject to unpredictable jump processes. Political shift could cause the economic program to' "reorganize" with little rhyme or reason. Less dramatic could be sudden breaks in the economic program that materialize when novel states of the world are encountered. For example, when stocks fall below an arbitrary threshold society may adopt a different management program. This is not fully separable from the potential stochastic political events. While it may be possible *conceptually* to translate these profound uncertainties into the distributional language of risk, this process may be infeasible in practice. These "unknown unknowns" will continue to make consensus on realized shadow prices challenging. Nevertheless, realized shadow prices play an important role in sustainability assessment. They help re-frame the resources that society is currently making and reveal the implied scarcity of societal assets given human actions.

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