Risky (Natural) Assets:
When Does Stochasticity Matter for Valuing Natural Capital?

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Abstract

The treatment of risk in respect to natural capital has focused on its implications for optimal management, with little attention being payed to the question of how risk alters the *valuation* of natural assets. We extend the theory of natural capital valuation in non-optimized systems to account for the effects of stochasticity in the dynamics of the natural capital, showing how risk alters the marginal valuation of natural capital through an "endogenous risk" effect and through an "endogenous risk aversion" effect that depend in complex ways on the properties of the welfare function from instantaneous consumption of natural capital, the properties of the natural capital dynamics, the stochastic process, and the economic program linking capital stocks and their human use. We also derive a formula showing how sub-optimal management influences the valuation of natural capital. Using canonical single-stock examples we demonstrate that in some cases these two risk effects may approximately balance so that risk has little effect on valuation. However, we demonstrate that risk can have large and complex effects on valuation when resource dynamics involve non-convexities.
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1. Introduction

Risk and uncertainty are a longstanding concern in natural resource economics and real-world resource management. Most research in this domain focuses on the management implications of risk for decision making – how the presence of risk or uncertainty alters the optimal control rule. Accordingly, there is a large literature applying stochastic optimal control theory to the optimal management of resources subject to stochastic shocks (LaRiviere et al., 2017). Tools from robust control are increasingly used to address decision making under Knightian model uncertainty. (e.g., Rodriguez et al., 2011; Roseta-Palma and Xepapadeas, 2004). Meanwhile, decision makers are influenced by a wave of thought, loosely organized under the heading of the "precautionary principle," urging less aggressive action, or the delay of irreversible actions, under conditions of risk or Knightian uncertainty. This mode of thinking is provided some qualified economic support by the literature on option value (Arrow and Fisher, 1974; Dixit and Pindyck, 1994; Gollier, 2003), but is countered by the literature on adaptive management (e.g., Walters, 1986), which urges active learning in the presence of risk and uncertainty. Given these divergent approaches, and their often conflicting advice, it is of little surprise that resource governance continues to struggle with how to incorporate risk and uncertainty into decision making.

Lost in the focus on the positive implications of risk and uncertainty for resource management is the question of how risk and uncertainty influence the *valuation* of natural capital. This is a distinct concern from pricing risk (or risk reduction) itself, as in the literature on quasi-option value or the value of information (Hanemann, 1989). Rather than pricing increments of risk, we are asking, how does the introduction of risk, or change in the magnitude of risk, alter the value attached to changes in the stocks of biotic or abiotic resources? In what circumstances might risk act to devalue capital stocks, and when might risk enhance a stock's value? How do the presence of non-linear dynamics and the presence of alternative stable states (Dasgupta and Mäler, 2004) alter the valuation of risk?

Addressing these questions is important for the prospective valuation of policy changes that alter resource stocks as well as for retrospective valuation of changes in these stocks, as in natural resource damage assessment. Valuation of natural capital is also critical for development of wealth-based indicators of sustainability (e.g., Dasgupta, 2001; Dasgupta and Mäler, 2000; Hamilton and Clemens, 1999) that have become increasingly common at the level of international and national assessments (UNU-IHDP and UNEP, 2014; Dasgupta, 2021) [1] and that have been used to assess the sustainability of bounded systems such as cities (Dovern, Quaas, and Rickels, 2014), hydrological catchments (Pearson et al., 2013) and as an indicator of sustainable management for ecosystems (Yun et al., 2017). Nevertheless, despite the importance of understanding the effects of risk on the valuation of natural capital, little research on this topic exists, at least within the natural resource economics literature.[2]

To understand the effects of risk on the valuation of natural capital, we must first situate risk within the theory of capital valuation (Jorgenson, 1963). Valuing biotic and abiotic resources as capital requires that they are valued in terms of the discounted present value of the service flows that derive from their consumptive or non-consumptive use – usually in combination with human and reproducible capital. Therefore, it is imperative that changes in natural capital stocks be mapped to changes in the trajectory of capital stocks and any associated service flows (Fenichel, Abbott, and Yun, 2018). In other words, valuation of natural capital must implicitly or explicitly be dependent on a *forecast* of the value of the service flows from capital stocks as well as their stocks themselves, where fully coupled human-natural systems models (e.g., bioeconomic models) provide one such forecast (Fenichel, Abbott, and Yun, 2018). An essential aspect of these forecasts is a description of how human behavior that influences service flows or the trajectory of capital stocks will change in response to the stocks of capital and other time-varying states of the system. Just as any financial analyst's

[1] For example, Canada contracted for a Comprehensive Wealth report in 2018 https://www.iisd.org/library/comprehensive-wealth-canada-2018-measuring-what-matters-long-term and the U.K. has developed a 25 year plan focused on natural capital, https://www.gov.uk/government/groups/natural-capital-committee.

[2] Kvamsdal et al. (2020) offer an analysis of "ocean wealth" in the Barents sea for a three-species ecosystem that includes stochasticity. However, the role of risk is ancillary to the multi-species focus of the analysis.

valuation of a company is contingent on their (often implicit) beliefs of how the company's management will leverage a company's capital stocks to provide valuable goods and services and then invest in these capital stocks moving forward, so any valuation of natural capital is contingent upon beliefs about the feedback control rule (i.e. the "economic program") linking capital stocks to the human behavior that "manages" the system. This includes expectations about how the manager will respond to risk and uncertainty. Yet, focus within bioeconomics has often centered upon the special case of control rules that optimize the present value of the system. While understandable from the perspective of advising policy, the fixation on optimal control rules fails to provide a realistic forecast of human interactions with natural systems in many cases. Given that many natural capital stocks provide service flows that are non-excludable, non-rivalrous, and are themselves often subject to incomplete property rights, collective action dilemmas, and other governance failures, it is not surprising that their management is often demonstrably inefficient, even 'kakatopic' (Dasgupta, 2001). Therefore, bioeconomic models that utilize optimality assumptions may provide a poor forecast of the management of natural capital stocks, thereby yielding questionable valuations of natural capital - valuations that are critical for credible sustainability assessments.

Fortunately, substantial theoretical, and some empirical, progress has been made in recent years in capital-theoretic valuation of natural resources in non-optimized settings. Drawing upon earlier work (e.g., Dasgupta and Mäler, 2000; Arrow et al., 2012), Fenichel and Abbott (2014) provide a theoretical foundation for pricing of natural assets by deriving the revealed shadow price or accounting price of natural capital under general, non-optimized forms of management, and link their derivation to foundational contributions in economic capital theory (Jorgenson, 1963).[3] In this and subsequent work with coauthors, they demonstrate the

[3]See Fenichel, Abbott, and Yun (2018) for a detailed development of natural capital pricing. This approach has subsequently been expanded to allow for the valuation of a portfolio of capital stocks whose dynamics may be interlinked through physical or biological processes or via human behavior (Yun et al., 2017). These methods have been used to value a range of natural capital stocks, from fish in single-species fisheries (Fenichel and Abbott, 2014), groundwater (Fenichel et al., 2016; Addicott and Fenichel, 2019), coastal habitat (Bond, 2017), an assemblage of interacting fish stocks (Yun et al., 2017), threatened species in an ecosystem context (Maher et al., 2020).

necessary components of an accounting price for natural assets and develop and implement computational approaches to measure accounting prices. However, risk and uncertainty are notably absent from the theory and empirical work springing from this literature. This omission limits the ability to answer the question at the core of this paper – how does risk alter the valuation of natural capital when it is valued according to the principles of capital theory?

We make progress on this research question by, first, expanding the theory of natural capital valuation to embrace stochasticity. The resulting equations provide intuition into the ways that risk can influence the *total* value provided by one or more capital stocks and *marginal* valuation – the "shadow price." We also expand upon the computational approaches utilized in prior work to show how these valuations can be uncovered in the context of coupled bioeconomic models. Second, we deploy these approaches using simple examples from natural resource management, using optimized and non-optimized control rules to examine the qualitative and quantitative effects of risk on natural capital valuation. Intriguingly, we find that under a range of specifications and reasonable risk levels the effect of stochasticity on valuation is minimal – so minimal as to suggest risk can be safely ignored for valuation in similar applications. One commonality of these examples, however, is convexity, with a single stochastic steady state. Therefore, as our third contribution, we consider the implications of introducing stochasticity to systems with non-convexities and multiple stable states by examining total and marginal valuations for a canonical non-convex ecosystem – a renewable resource with a minimum viable population – under stochasticity. We find that non-convexity interacts with stochasticity in complex ways – potentially reversing the finding that risk "doesn't matter" for valuation while also showing that the effects of risk on marginal valuation of natural capital are highly contingent on the stock level relative to critical threshold locations in the state space. In short, risk can depreciate investments in natural capital at some stock levels and raise the value of investments at others.

The organization of the paper is as follows. The following section derives the shadow price formulas for the single- and multi-stock cases. Section 3 demonstrates the valuation approach for a canonical, single-stock, stochastic control problem where the optimal co-state (i.e. accounting price) is available in closed form. This allows us to validate our approach and also allows us to isolate the effects of stochasticity from the effects of the choice of a sub-optimal control rule (economic programs) that may stem from heuristics for coping with stochasticity. Section 4 extends our approach to a stochastic version of the Gulf of Mexico reef fish case study examined in Fenichel and Abbott (2014). This case allows us to consider the impact of natural stochasticity in a real-world, non-optimal setting under conditions of convexity. Section 5 extends our analysis to non-convex settings, using the case of a harvested resource with a minimum viable population as an example. Section 6 concludes the paper.

2. Theory

2.1. Derivation of shadow pricing formula

Let $s(t)$ represent the known stock of a scalar asset at time t.[4] Suppose the dynamics of s are represented by a diffusion or Ito process with stationary infinitesimal parameters $\mu\left(s, x\left(s\right)\right)$ and $\sigma(s)$. The diffusion process is written as

$$ds(t) = \mu\left(s(t), x\left(s(t)\right)\right) dt + \sigma(s(t)) dZ(t) \tag{1}$$

where $dZ(t)$ is an increment of a Wiener process (Stokey, 2009). The drift of the diffusion $\mu\left(s, x\left(s\right)\right)$ is specified as a function of the current capital stock and as a function of the feedback control rule, known as the economic program or resource allocation mechanism, $x(s)$. Once the substitution for the economic program has been made, the drift is an explicit function of only s.

Let $W(s, x)$ be an instantaneous measure of welfare. Importantly, the concavity prop-

[4] t is suppressed when doing so does not cause confusion.

erties of this function are quite general, allowing for the full range of risk preferences in "consumption" (x) and the resource stock. Define the intertemporal welfare function, evaluated along the economic program and along the stochastic capital trajectory given by (1), as

$$V\left(s\left(t\right)\right)=\mathbb{E}_t\left[\int_t^{\infty}e^{-\delta(\tau-t)}W\left(s\left(\tau\right),x\left(s\left(\tau\right)\right)\right)d\tau\right] \tag{2}$$

where $\mathbb{E}_t$ is the expectations operator. The marginal value of an investment in the capital stock in expectation is defined as $p(s)\equiv V_s$. To derive the properties of $p(s)$, start by differentiating (2) with respect to t.

$$\frac{dV}{dt}=\mathbb{E}_t\left[\delta\int_t^{\infty}e^{-\delta(\tau-t)}W\left(\cdot\right)d\tau-W\left(s\left(t\right),x\left(s\left(t\right)\right)\right)\right]=\delta V-W\left(s\left(t\right),x\left(s\left(t\right)\right)\right) \tag{3}$$

The first equality in (3) assumes that the derivative can be carried through the expectation operator, which is ensured by the stationarity of the infinitesimal parameters of (1). The second equality holds because the state of the system is known at $\tau=t$.

We know that $\frac{dV}{dt}=\frac{\mathbb{E}_t[dV]}{dt}$. By Ito's Lemma

$$dV=\left[\mu\left(s\right)V_s+\frac{1}{2}\sigma^2\left(s\right)V_{ss}\right]dt+\sigma\left(s\right)V_sdZ$$

Taking the expected value, and employing the property that all stochastic integrals are identically zero (Stokey, 2009):

$$\mathbb{E}_t[dV]=\left[\mu\left(s\right)V_s+\frac{1}{2}\sigma^2\left(s\right)V_{ss}\right]dt$$

so that

$$\frac{dV}{dt}=\frac{\mathbb{E}_t[dV]}{dt}=\mu\left(s\right)V_s+\frac{1}{2}\sigma^2\left(s\right)V_{ss} \tag{4}$$

Setting (3) equal to (4) we obtain the stochastic Hamilton-Jacobi-Bellman (HJB) equation:

$$\delta V\left(s\right)=W\left(s\left(t\right),x\left(s\left(t\right)\right)\right)+\mu\left(s\right)V_{s}+\frac{1}{2}\sigma^{2}\left(s\right)V_{ss} \tag{5}$$

Substitute $p(s) \equiv V_S$ into the HJB equation yielding:

$$\delta V\left(s\right)=W\left(s\left(t\right),x\left(s\left(t\right)\right)\right)+p\left(s\right)\mu\left(s\right)+\frac{1}{2}\sigma^{2}\left(s\right)p_{s}\left(s\right) \tag{6}$$

The first two terms on the RHS are the traditional deterministic current-value Hamiltonian. The third term captures the effect of risk even if the deterministic rate of change in the capital stock $\mu(s) = 0$. The risk effect captures the effect of Jensen's inequality via the curvature of the intertemporal welfare function. If the shadow price function is downward sloping then $p_s < 0$ so that risk has a negative effect on the intertemporal welfare function.

Importantly, the derivatives V_s and V_{ss} in (5) (i.e. p and p_s in (6)) are evaluated after substitution of a particular feedback control rule $x(s)$, which in general need not optimize the HJB. The implication is that the slope, curvature, and higher derivatives of the intertemporal welfare function reflect, in part, the properties of this economic program. The one exception to this case is when the economic program is dynamically optimal. In this case the properties of the economic program play no role in the shape of the intertemporal welfare function.[5]

Suppressing functional dependency on s, and differentiating (6) with respect to s yields:

$$\delta p=W_{s}+\mu_{s}p+\mu p_{s}+\sigma\sigma_{s}p_{s}+\frac{1}{2}\sigma^{2}p_{ss}$$

Isolating p on the left-hand side we obtain the asset pricing equation:

$$p(s)=\frac{W_{s}+[\mu(s)+\sigma(s)\sigma_{s}(s)]p_{s}+\frac{1}{2}\sigma^{2}(s)p_{ss}}{\delta-\mu_{s}(s)} \tag{7}$$

A parallel derivation of this pricing equation for the multi-stock case is provided in

[5]See Section 2.3 for the mathematical and economic justification of this assertion.

Appendix A.

2.2. Intuition

In the case where the variance of the noise in (1) does not depend on s then (7) reduces to:

$$p(s) = \frac{W_s + \mu(s)p_s + \frac{1}{2}\sigma^2 p_{ss}}{\delta - \mu_s(s)}$$

and if capital dynamics are deterministic or $p_{ss} = 0$ then this further reduces to

$$p(s) = \frac{W_s + \mu(s)p_s}{\delta - \mu_s(s)}$$

which is the same as in Fenichel and Abbott (2014), who show that this equation is equivalent to Jorgenson (1963). The general asset pricing equation equation (7) contains two additional numerator terms relative to Fenichel and Abbott's deterministic derivation. The first term enters in a way that is symmetric to capital gains in a deterministic system and depends on the extent of "risk aversion" embodied in the curvature of the intertemporal welfare function (since $p_s \equiv V_{ss}$) and the extent to which the standard deviation of the diffusion is elastic with respect to s. If increasing investment in s increases the size of shocks, and if the shadow price function is decreasing in the stock (analogous to risk aversion), then this results in a "capital loss." This term only matters if the variance depends on the capital stock, as in the case of geometric Brownian motion. Importantly, curvature of the intertemporal welfare function, which is defined over the domain of the capital stock only, need not result from underlying curvature of the "social utility" or real income function for welfare flows $W(\cdot)$. Indeed, the nature of risk preferences over flows embodied in W (including risk neutrality) may have no direct mapping to the curvature of $V(s)$. Curvature of the intertemporal welfare function can also be inherited from the underlying biophysical dynamics in (1) or from non-optimal economic programs $x(s)$ – suggesting that the risk premia embodied in the numerator of (7) are endogenous to policy and may reflect actual existing levels of self-insurance and self-protection (Ehrlich and Becker, 1972). This first term pertains to how a marginal investment in the capital stock increases risk, holding the curvature of the intertemporal welfare function constant.

The second additional term in (7) is present with stochastic dynamics so long as the third derivative of the intertemporal welfare (or value) function is non-zero. There will be a premium if there is a positive third derivative (convex price function), while a negative third derivative (concave price function) yields a discount. If the value function is quadratic (i.e. zero derivatives above the second derivative), then this term is zero. Both additional terms in the numerator of (7) originate from differentiating $\frac{1}{2}\sigma^2(s)p_s$ term in (6). This second term can be interpreted as the affect of a marginal increase in the capital stock on risk aversion, holding risk constant or interpreted as "prudence," which is associated with precautionary savings (Kimball, 1990). If risk aversion is increased by the investment ($V_{sss} = p_{ss} < 0$) then the shadow price is decreased. In other words, the pricing of risk into the capital asset depends on how additional investment affects the sensitivity to risk, given the biophysical dynamics and economic program in place, in addition to how the marginal investment affects the risk itself. The former effect can be thought of as a "self insurance effect" because changes in the curvature of the intertemporal welfare function impact the consequences of stochastic events rather than their probability (Shogren and Crocker, 1999).

Importantly, (7) tells us that risk will have no *marginal* effect on the valuation of natural capital when two conditions hold: 1) the volatility of risk is not a function of the capital stock (i.e. risk is "exogenous"); and 2) the shadow price function is linear ($p_{ss} = 0$), so that the intertemporal welfare function is quadratic. This coincides with well-known results in economic theory that optimizing agents facing exogenous shocks to their income will not alter their saving behavior in the presence of a mean preserving spread in risk if they have quadratic utility in consumption. In this case (7) collapses to the co-state equation for the deterministic case, so that optimal behavior is invariant to risk.[6]

[6]The fact that these results are specified in terms of the third derivatives of the instantaneous utility function (in consumption) rather than the intertemporal welfare function (in unit of savings or capital) is because the linear properties of the growth function of capital in the simple savings problem, combined

2.3. Optimized vs. non-optimized asset pricing

In this section we demonstrate how non-optimal economic programs alter the marginal valuation of capital stocks, relative to the co-state evaluated along the optimal program. Let the non-optimal program for a given setup be $x(s)$, while the optimized program is $x^*(s)$. The intertemporal welfare function along the non-optimal economic program is $V(s, x(s)) = V(s)$. The shadow price can therefore be written as:

$$V_s = \frac{dV}{ds} = \frac{\partial V}{\partial s} + \frac{\partial V}{\partial x}\frac{dx}{ds} \tag{8}$$

By the maximum principle and as a consequence of the envelope theorem $\frac{\partial V}{\partial x} = 0$ when $x(s) = x^*(s)\ \forall s$. In this case (9) collapses to $V_s = \frac{dV}{ds} = \frac{\partial V}{\partial s}$. This suggests that the shadow price can be cleanly broken into two terms: 1) the partial derivative of the value function w.r.t. s, which is present in the optimized case; and 2) a component that is only present in non-optimized cases, and that therefore reflects a "non-optimality bias" of sorts. This is *almost* correct but neglects to account for the fact that $\frac{\partial V}{\partial s}$ is evaluated at the non-optimized value of x for a given s, whereas it is evaluated at the optimal x in the optimized case.

To account for this, add zero in the form of $\frac{\partial V(s,x^*(s))}{\partial s} - \frac{\partial V(s,x^*(s))}{\partial s} = \frac{\partial V^*(s)}{\partial s} - \frac{\partial V^*(s)}{\partial s}$ to (9)

$$V_s = \frac{\partial V^*(s)}{\partial s} + \left(\frac{\partial V}{\partial s} - \frac{\partial V^*(s)}{\partial s}\right) + \frac{\partial V}{\partial x}\frac{dx}{ds} \tag{9}$$

This equation tells us that we can write the shadow price as the optimal shadow price (which is, by definition, evaluated at the optimal value of the economic program) plus a first "distortion term" that is the gap in the partial derivative of the value function that arises solely through the gap between $x(s)$ and $x^*(s)$ plus a final distortion term that is a function of the sensitivity of the non-optimized intertemporal welfare function to the economic program $\frac{\partial V}{\partial x}$ and the slope of the non-optimized economic program itself.

with the selection of the *optimal* investment rule ensure that the shape properties of the utility function are reflected in the intertemporal welfare (value) function as well. This mapping does not carry over to the case of non-optimized control rules with non-linear drift terms, as in the case of most natural capital.

The first distortion arises only because the partial derivative of V is being evaluated at a level of x that does not set marginal benefits equal to marginal costs inclusive of the shadow price at a given s. If the optimal and non-optimal economic program coincide at a particular s then this term vanishes. For example, suppose that the optimized and non-optimized economic programs both achieve the same steady state so that $x(s_{SS}) = x^*(s_{SS})$, then this term would vanish at the steady state stock. In general, if $V_{sx} = 0$ then this distortion term is zero.

The second distortion term arises because of the slope of the non-optimal economic program, rather than its level. Note that the more non-optimal the economic program is – the larger $\frac{\partial V}{\partial x}$ is – the larger this will be. Also, the more responsive the economic program is to the state – the larger $\frac{dx}{ds}$ is in absolute terms – the larger this distortion will be. We may also be able to sign the distortion. If, at the non-optimal level of $x(s)$, it would improve intertemporal welfare to harvest less instantaneously and the economic program is upward sloping, then we know that the distortion is negative, causing sub-optimal management to lead to an undervaluation of the stock.

3. An optimized single-stock example

The asset pricing equation presented in the previous section is valid regardless of whether the economic program maximizes intertemporal welfare or not. Still, it is useful to build intuition for the role of risk in shadow prices from an optimized model. Simple optimized models may also confer the benefit of a closed form solution for the co-state, thereby allowing for a direct validation of the numerical approximation approach.[7]

To provide this example, we draw upon a case developed in Pindyck (1984). In this seminal contribution, Pindyck extends the canonical infinite horizon, continuous-time renewable resource model for a single stock to allow for a stochastically evolving resource stock. The

[7] Fenichel and Abbott (2014) follow a similar process in the deterministic case by evaluating how the natural asset pricing approach works on a simulated optimal program.

focus of the modeling is on revealing how the 'golden rule' of resource management is augmented by a risk premium term. Pindyck then explores how the biological and economic parameterization interacts with increases in risk to influence the optimal extraction rate and the stochastic steady state distribution.

Our model draws directly on Pindyck's example 1 (p. 296), which is also explored in Miranda and Fackler (2004, p. 330). Pindyck's objective is to maximize the expected net present value of the combined consumer and producer surplus from harvest q of the fish stock s over an infinite horizon, where the demand function is isoelastic, $q(p) = bp^{-\eta}$, and the marginal cost of harvest is $cs^{-\gamma}$.[8] Therefore, the social planner is risk-neutral in their assessment of flows of monetized instantaneous social welfare. Note, however, that these assumptions nevertheless imply that the instantaneous welfare function is strictly concave in harvest, implying aversion to any risk-induced fluctuations in harvest levels. The resource dynamics evolve according to a diffusion characterized by a logistic drift function with stochasticity that follows a geometric Brownian motion process: $ds = [rs\,(1 - s/K) - q]\,dt + \sigma s\,dZ$.

In general, this model must be solved numerically. However, Pindyck (1984) demonstrates that a closed form solution to the HJB equation exists when $\eta = 1/2$ and $\gamma = 2$. Specifically, the optimized co-state (or rent) is:

$$V_s = \phi/s^2 \tag{10}$$

and the optimized economic program (feedback control rule) is:

$$x(s) = q^*(s) = \frac{b}{(\phi + c)^{1/2}} s \tag{11}$$

where

$$\phi = \frac{2b^2 + 2b[b^2 + c(r + \delta - \sigma^2)^2]^{1/2}}{(r + \delta - \sigma^2)^2} \tag{12}$$

[8] Pindyck uses x for the state variable, we have changed this to s to avoid confusion and align with notation within this paper.

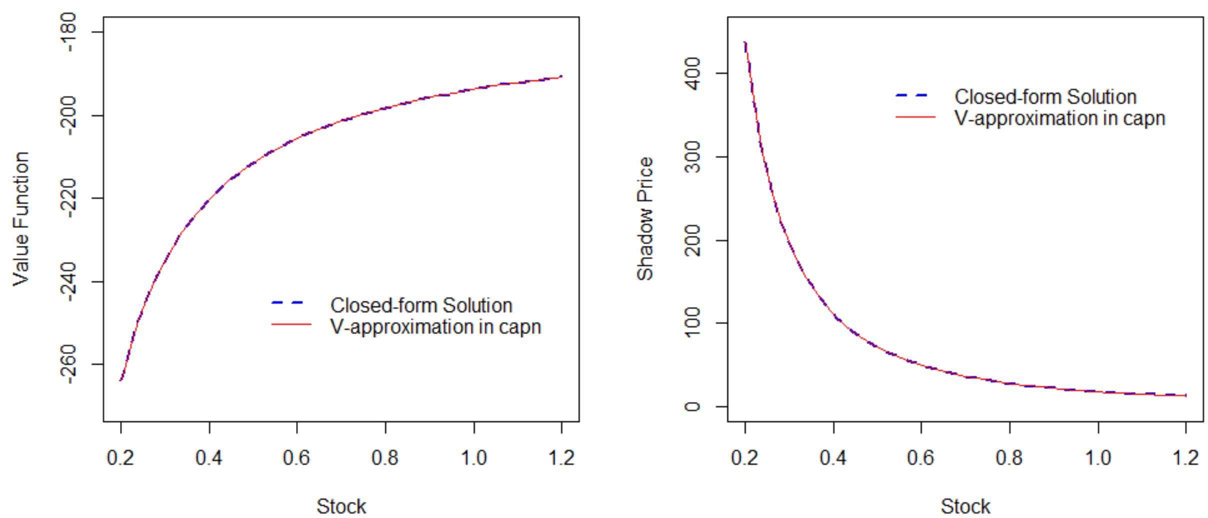


Figure 1: Illustration the the natural capital asset pricing approximation approach reproduces known value function and price curves for a stochastic system.

The resulting economic program (11) increases linearly in the stock. Such rules imply a constant (per-capita) rate of fishing mortality – what the fisheries literature calls a "constant-F" rule. This is a common harvest rule in natural resource management (Deroba and Bence, 2008). Importantly, $\partial\phi/\partial\sigma^2 > 0$. This implies that $\partial x/\partial x = \partial q^*/\partial\sigma^2 < 0$ and $\partial V_s/\partial\sigma^2 > 0$, meaning that increasing stochasticity in this model always increases the accounting price of the stock, thereby *decreasing* the optimal rate of harvest at every stock level.[9]

We approximate the value function using an extension of the basis function approximation approach employed in Yun et al. (2017) as detailed in Appendix B.[10] Figure 1 shows that we are able to reproduce the analytical value function and shadow price to a high degree of accuracy.

Dynamic optimization in this example yields an economic program that reflects what we might term "uniform precaution" (Figure 2, black line). Increases in stochasticity lead to

[9]The partial derivative of the economic program with respect to σ^2 is a comparative static – the volatility parameter is constant. This means that in a sense the risk profile is not changing over time or the system is not becoming more or less risky.

[10]We use parameter values of $\sigma = 0.1, \delta = 0.05$, $b = 1$, $r = 0.5$, and $K = 1$.

a less aggressive harvest rate at *all* stock levels and therefore a larger 'target' steady state biomass. Pindyck shows that stock stochasticity has three competing effects in an optimized stochastic problem that may lead to more or less aggressive (less or more precautionary) harvest relative to the deterministic case. The first, a variance reduction effect, encourages the manager to hold a smaller stock due to the fact that the variance of stock increases in the stock size, and variance lowers the value function, which follows from the concavity of value function. This effect works through the first risk-specific numerator term in (7). The second, a cost reduction effect, encourages the manager to hold a smaller stock as variance increases due to the cost-increasing effects of stochastic fluctuations on expected harvest costs because of the concavity of the harvest cost function – an implication of Jensen's inequality. Third, a growth rate effect encourages managers to hold *more* stock as variance rises because stochasticity reduces the expected growth rate of the stock, which follows from the concavity of the growth function. Both the cost reduction and growth rate effects operate through the second risk-specific numerator term in (7). Pindyck shows through a series of examples how the different effects can lead to more or less aggressive harvest, and hence higher or lower valuations, relative to the deterministic case, under risk. Of particular importance are the elasticity of demand and the skewness of the growth function.

In practice, managers may choose to exercise more (or less) precaution than the level that maximizes Pindyck's object. We reflect these adjustments through two scalar shifts of the economic program, where harvests are either systematically lower (purple line, half the optimal harvest at every point) or greater (red line, 1.5 times the optimal harvest at every point) than the optimizing program (black line) (Figure 2). These shifts lead to economic programs that are non-optimal everywhere and result in different stochastic equilibria. These deviations from optimality are reflected in the intertemporal welfare functions and accounting price functions.

We also consider an economic program that deviates from the "constant-F" form (Figure 2, blue curve) by being a convex function of the stock. This program is 'adaptive' in its

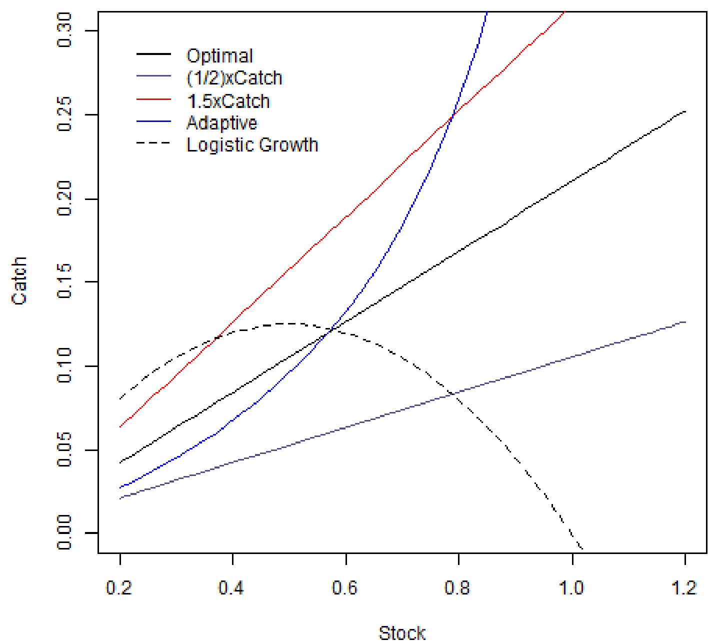


Figure 2: Stock-catch space showing the optimal harvest feedback rule and three alternative non-optimal economic programs

degree of precaution by being more conservative at low stocks and more aggressive at high stocks.[11] For the sake of comparison, we calibrate this control rule to have the same stochastic equilibrium as the optimal program. Therefore, the adaptive program is the optimal program if, and only if, the stock is at the stochastic equilibrium. The strong convexity of the adaptive program reflects an important feature of real natural resource management systems – managerial bias for system stability. In this case, the steady state probability distribution will have a lower variance than the optimal program. The shape of this control rule, but not its anchoring on the optimal steady state, is similar to the feedback process that Zhang and Smith (2011) estimate and Fenichel and Abbott (2014) use in their application to the Gulf of Mexico reef fish fishery.

Figure 3 compares the intertemporal welfare and price functions for the scalar trans-

[11]In fisheries management, this adaptivity is often accomplished in practice by distinct linear harvest control rules that are each applicable within different stock thresholds–in essence a linear spline function.

formations of the optimal harvest program for the stochastic ($\sigma = 0.1$, solid lines) and deterministic case ($\sigma = 0$, dotted lines). Importantly, the optimal and sub-optimal economic programs adjust for the value of σ according to the feedback rule in (11). The left-hand panel, showing the intertemporal welfare functions, shows that risk strictly reduces welfare. Risk has a similar effect across all three economic programs. Stochasticity appears to translate the intertemporal welfare functions down in a nearly constant manner (i.e. a location shift). This suggests that *changes* in welfare between stock levels – which are the relevant metrics for social benefit cost analysis and sustainability assessment – may be minimally affected by volatile stock dynamics. The first column of Table 1 considers the welfare change for a relatively large perturbation in stock from 0.37 to 0.57. Regardless of whether we consider the optimal or sub-optimal programs, we find that the change in welfare from a stock shift is 3 percent greater in the stochastic case relative to the deterministic case, despite substantial volatility. Therefore, ignoring stochasticity may systematically undervalue changes in natural capital. However, our example indicates that this bias may be small in some cases.[12] Indeed, we find that the changes in measured welfare across the three economic programs – holding stochasticity constant – are much more sizable than the effects of ignoring stochasticity. This suggests that the *behavioral or managerial responses* to stochasticity (i.e., excessive or inadequate precaution that push the system toward a sub-optimal equilibrium) may be more consequential for welfare than the effects of stochasticity itself.

Given the apparently near-vertical translations of the intertemporal welfare functions from introducing stochasticity, it unsurprising that risk has a muted effect on the accounting price functions (Figure 3). This is a direct effect of the price being the first derivative of the value function. The effect of stochasticity on the shadow price is hardly noticeable. For the optimal program and its scalar multiples, price always increases in stochasticity. However,

[12] We find that employing the program associated with deterministic dynamics to a system with stochastic dynamics has a small effect. Using the "wrong" program can either lead to a larger or small assessment of the welfare change, relative to using the stochastic program. This is due to second-best nature of these feedback rules.

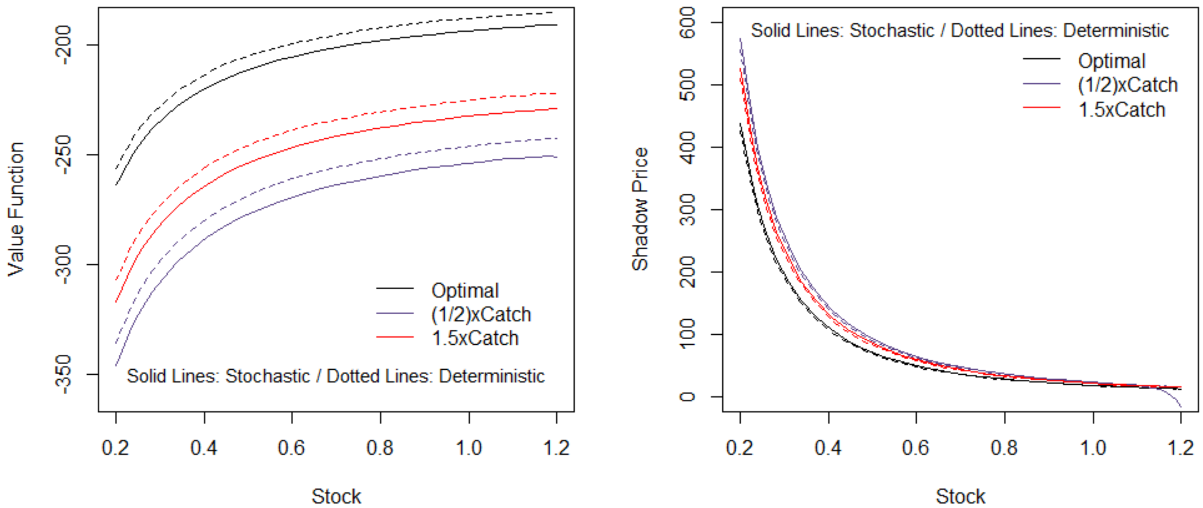


Figure 3: The intertemporal welfare (value) function and shadow price curves for optimal program and two scalar shifts of the optimal program with stochastic and deterministic dynamics.

stochasticity has only a second-order effect on marginal values – hardly surprising given the small welfare effects of stochasticity for non-marginal stock changes (which integrate under the price curve) noted in the previous paragraph. [13]

Now consider the adaptive economic program (Figure 4), which mimics the asymmetric precaution observed in the management of many harvested resource systems. This added precaution ensures a greater degree of stability relative to linear feedback rules.[14] Importantly, this rule has the same steady state as the optimal control rule, so all differences are due to the sub-optimal approach path and its potential interactions with stochasticity. As before, the most apparent effect of introducing stochasticity to the adaptive economic program is a downward shift in the intertemporal welfare function. However, there are subtle, but important differences relative to the optimal (linear) control rule case.

In the deterministic case (dashed curves), the intertemporal welfare value in the region

[13]Stochasticity has no effect on the approximation error of welfare changes introduced by using a price index over the change multiplied by the change in quantity (Table 1). The use of price indexes is likely necessary in applied wealth accounting approaches for sustainability assessment. The Fisher Ideal price Index is the geometric mean of prices. The Mean price Index is the arithmetic mean of prices.

[14]This may suggest managerial risk aversion or unobserved adjustment costs.

Table 1: Comparison of the change in welfare and the change in wealth using two different index number approaches. The Fisher Ideal index is the geometric mean of prices, and the Mean price index is the arithmetic mean of prices. The price index is multiplied by the change in quantity.

Program	Change in Welfare	Fisher Ideal Index	%error	Mean Price Index	%error
Optimal rule with deterministic dynamics	15.920	15.920	0.000	17.373	0.091
Optimal rule with stochastic dynamics	16.405	16.405	0.000	17.902	0.091
Adaptive rule with deterministic dynamics	17.110	16.901	-0.012	18.873	0.103
Adaptive rule with stochastic dynamics	17.730	17.517	-0.012	19.553	0.103
Scalar rule, 0.5 of the optimum, with deterministic dynamics	20.855	20.855	0.000	22.758	0.091
Scalar rule, 0.5 of the optimum, with stochastic dynamics	21.497	21.497	0.000	23.459	0.091
Scalar rule, 1.5 of the optimum, with deterministic dynamics	19.081	19.081	0.000	20.822	0.091
Scalar rule, 1.5 of the optimum, with stochastic dynamics	19.692	19.692	0.000	21.489	0.091

of the equilibrium is approximately the same under the adaptive and optimal programs (indeed, identical at the equilibrium itself). Therefore, small changes in the stock in this region result in near-identical welfare changes under either program. It is only as the system moves significantly from the equilibrium that there is a meaningful divergence between the intertemporal welfare functions in a deterministic system. By contrast, when the system is stochastic, the intertemporal welfare function under the adaptive program is always below that of the optimal program – even at the stochastic steady state biomass. This occurs because even at the equilibrium point there is an expectation of a shock that will move the system to a region where the adaptive program is meaningfully sub-optimal.

These features are reflected in the shadow price curves (Figure 4, right panel). The

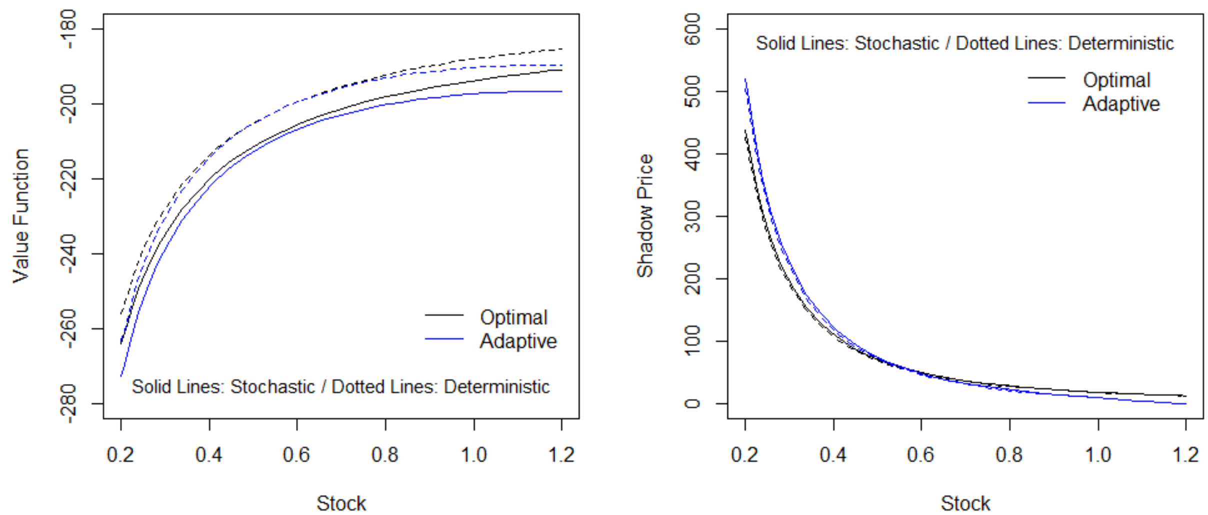


Figure 4: The intertemporal welfare (value) function and shadow price curves for the optimal program and a non-optimal adaptive "precautionary" economic program that preserves the stochastic equilibrium under stochastic and deterministic dynamics.

shadow price curves cross at the equilibrium. This must be the case in the deterministic model for the adaptive program to be sub-optimal everywhere except at the equilibrium; it is required for the intertemporal welfare function of the adaptive program to "bow in" relative to the optimal program's value function. This feature is inherited in the stochastic setting as well.

Despite these subtleties, the shadow price curves for the deterministic and stochastic dynamics for the adaptive precautionary economic program remain remarkably similar (2.7 % mean absolute error). Once again, the first order effects for valuation derive from the choice of the economic program – not from the introduction of stochasticity.

The analysis of Pindyck's model suggests that stochasticity may be, at most, a second-order concern for social benefit-cost analysis or sustainability assessment in some settings.[15] This appears in sharp contrast to much of the literature's broader concern with stochasticity,

[15]Lest the reader think we are cherry-picking an extreme example to minimize stochasticity, Pindyck's example 2 in the same paper replaces the logistic growth function with a Gompertz growth function to show that risk has *no* effect on shadow prices, and hence *no effect on changes in welfare*, in the optimal management case.

risk, and uncertainty. In the next section, we investigate a real-world system that that has similar features to the Pindyck model to see if these findings are robust to real-world, non-optimized management.

4. Gulf of Mexico (GOM) Reef Fish

The Gulf of Mexico reef fish example presented in Fenichel and Abbott (2014) has many of the same properties as the Pindyck (1984) model. Zhang and Smith (2011) estimated a logistic growth equation for the stock, and Zhang (2011) estimated an empirically-grounded feedback rule with similar properties as the adaptive rule illustrated in prior section – though Zhang's rule is not calibrated to achieve the optimal equilibrium (Figure 5).

We extend this deterministic model to the stochastic case while maintaining all other calibrated values. As in Pindyck, we augment the logistic stock dynamics with an additive geometric Brownian motion (GBM) noise term. Geometric Brownian motion is consistent with the assumptions of log-normal disturbances frequently used in population dynamic modeling and fisheries stock assessment. Utilizing the assessed biomass data from the fishery we calibrate $\sigma = 0.067$; therefore the standard deviation from the deterministic drift given by the logistic growth equation with harvest is approximately 6.7 percent of the stock level. The stock dynamics are:

$$ds = \left(rs(t)\left(1 - \frac{s(t)}{K}\right) - h\left(x\left(s\left(t\right)\right), s\left(t\right)\right)\right)dt + \sigma s(t)dZ(t), \tag{13}$$

where the intrinsic growth rate $r = 0.3847$ and carrying capacity $K = 3.59 \times 10^8$. The economic program, the feedback relationship linking stock status (in pounds (lbs)) and effort (in crew-days) in the fishery, is provided by a power rule, $x(s) = ys^{\gamma}$, where $\gamma = 0.7882$ and $y = 0.157$. We assume that the valuation of income flows in the fishery is directly expressed in terms of monetary profits, with price-taking firms and costs that are linear in effort: $W = mh - cx$, with $m =$ \$2.70/lb., $c =$ \$153/crew-day. Thus instantaneous welfare is "risk-neutral" in monetary flows. The production function for harvests is of a generalized

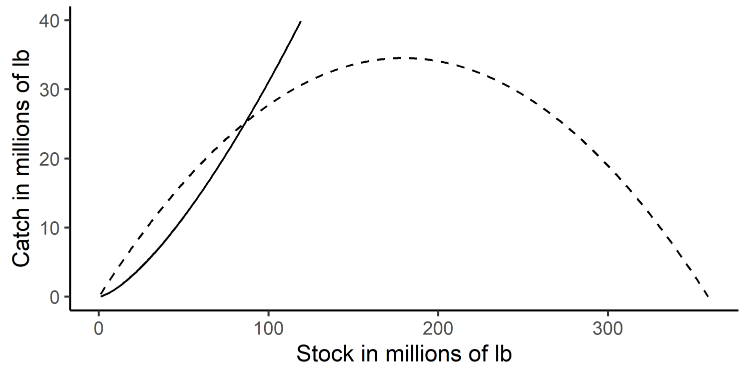


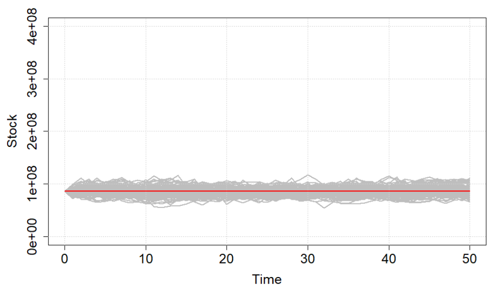
Figure 5: The growth function and economic program for the Gulf of Mexico model.

Schaefer form $h = qsx\,(s)^{\alpha}$, with $q = 3.17 \times 10^{-4}$ and $\alpha = 0.544$. $W(s)$ is a strictly *convex* function of the stock once the endogenous feedback from the stock level to harvest behavior $x(s)$ is incorporated, despite the linearity of harvests and costs for a fixed allocation of effort x. Abstracting from stochasticity, Figure 5 shows the dynamics of the system are similar to the Pindyck model with the adaptive control rule.

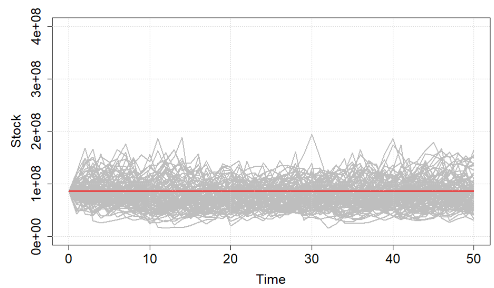
4.1. Stock Dynamics

Figure 6 illustrates stochastic simulations of stock paths originating from the steady state biomass and harvest under four levels of stochasticity. The level of noise introduced by stochasticity in the base case (Fig. 6a) is reminiscent of the noise seen in many ecological systems. While extinction is technically impossible in continuous time, using the current economic program and geometric Brownian motion, our numerical simulations nevertheless show that the number of paths that tend to a numerically zero level increase dramatically with increases in σ. Indeed, all paths reach numerical extinction within 20 periods when $\sigma = 1.0$. This suggests that levels of σ of 0.5 or 1.0 are likely inconsistent with the dynamics of most real-world species. Dixit and Pindyck (1994) provide a similar example where they argue that volatility can only be so high given a reasonable probability of observing the stock at all.[16]

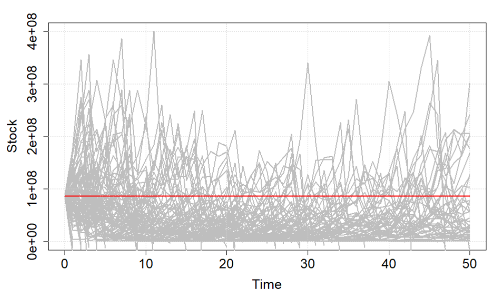
[16] We thank Martin Quaas for bring this to our attention.



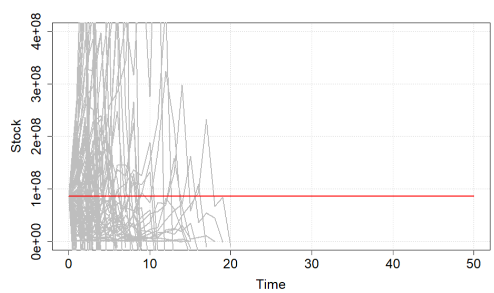
(a) $\sigma = .067$



(b) $\sigma = .2$



(c) $\sigma = .5$



(d) $\sigma = 1$

Figure 6: Stochastic simulations of stock dynamics over a range of values for σ. Note that the values for $\sigma = 1$ exceed the range of the graph on a number of runs.

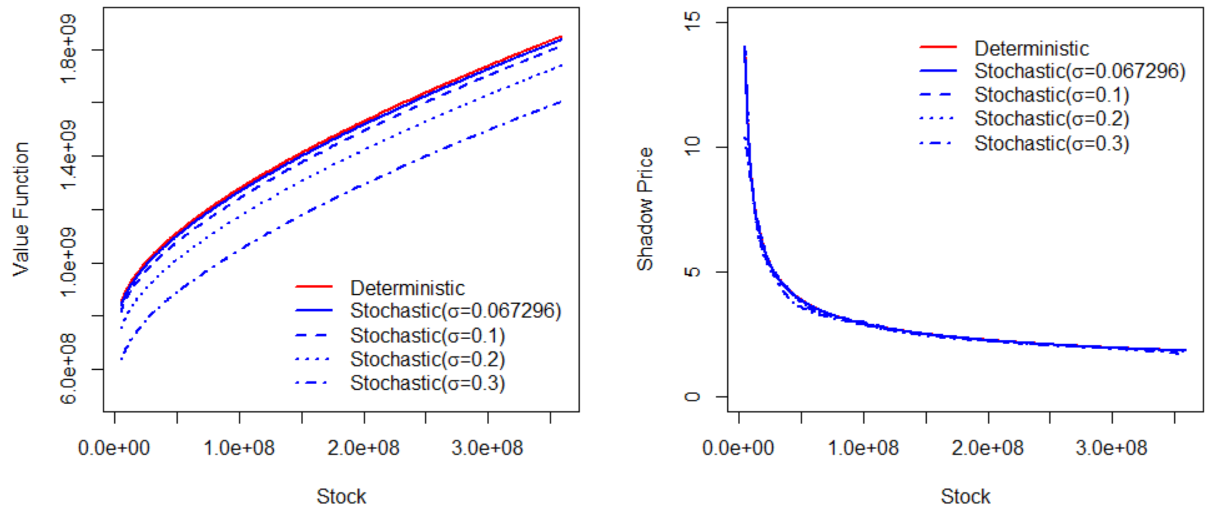


Figure 7: The intertemporal welfare or value function and shadow price or account price function of the Gulf of Mexico reef fish example with four different values of σ

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4.2. Value Function and Shadow Price

Unsurprisingly, increasing levels of volatility reduce the intertemporal welfare shown by the graph of the value function (Figure 7). Following Eq. (6) and the Pindyck example, the value function in the stochastic case includes an additional risk term that serves, in part, to shift the value function downward with increasing stochasticity. In the current case, the value function is concave in s (i.e. the shadow price curve is downward-sloping) so that increasing σ reduces the expected net present value at any given stock level. This adjustment is small (% error for price curves to the deterministic is 0.1, 0.9, 1.6, and 4.6 for each σ) for the empirically-justified, and objectively fairly large, level of stochasticity in our system ($\sigma = .067$) – suggesting that the economic program is fairly robust to the level of stochasticity in the system by maintaining stock levels in a relatively insensitive range of the profit function.

Higher-order effects on the shape of the value function with increases in σ exist, but are small. Changes in the shadow prices (i.e. the derivative of the value function) are hardly noticeable (Fig. 7, right panel) and suggest the volatility is creating a nearly vertical shift in the value function. Thus, while risk devalues the stock in total (albeit mildly), stochasticity has little appreciable effect on its *marginal* valuation. Lost in the small magnitude of these changes is a potentially interesting insight; whereas in the Pindyck example, increasing risk increases the marginal valuation of the stock, in this case risk *lowers* the shadow price.

Consider the value of a change from the observed equilibrium to the stock level supporting maximum sustained yield or half of carrying capacity. The change in the value function for the deterministic case is $244 million, whereas in the stochastic case the value is $243 million.[17] As expected, given the aforementioned effect of risk on marginal valuations, use of the deterministic system as a proxy for the stochastic system appears to overvalue the change in welfare or wealth slightly. This differs from Pindyck's optimized example, where failure to consider risk leads to a slight *undervaluation* of the value of stock changes. Whether under- or overvaluation is more likely ultimately depends upon the bioeconomic parameters.

On the whole, the Gulf of Mexico case reinforces the intuition developed by the Pindyck example in the context of a real world, well-calibrated system governed by non-optimized control rules. Risk appears to be a decidedly second-order feature for achieving accurate valuations of either marginal or non-marginal changes in natural capital stocks. We conjecture that a key feature of both the Pindyck model and the GOM example that support this result is the existence of a single stochastic equilibrium. The next section develops a simple counterexample to explore the implications of thresholds and alternative stable states on natural capital valuation.

5. GOM Reef Fish with Minimum Viable Population

There is a substantial literature on multiple equilibria (reviewed by Fenichel et al. (2015)), but this literature largely focuses on deterministic models to examine how the optimal pursuit of alternative long-run equilibria depends on initial conditions. Fenichel, Abbott, and Yun

[17] Using the Fisher Ideal index the change in value for the deterministic and stochastic cases are both $245 million and using a mean price index both are $250 million. In both cases there are difference of less than $1 million.

(2018) argue that the mathematical difficulties caused by multiple equilibria for valuation purposes – where the economic program is typically pre-determined, and the relevant basin(s) of attraction are therefore known – are lessened compared to optimally controlling a system in the presence of non-convexities. However, introducing stochasticity to such a system brings with it the possibility of the system probabilistically entering or leaving one basin of attraction for another at a rate that is endogenous to the place in state space. The qualitative and quantitative effects on the value of the capital stock are potentially complex and poorly understood. As a simple example, we extend the Gulf of Mexico reef fish example to consider a canonical example of non-convexity, a harvested system under critical depensation.

5.1. Stock Dynamics

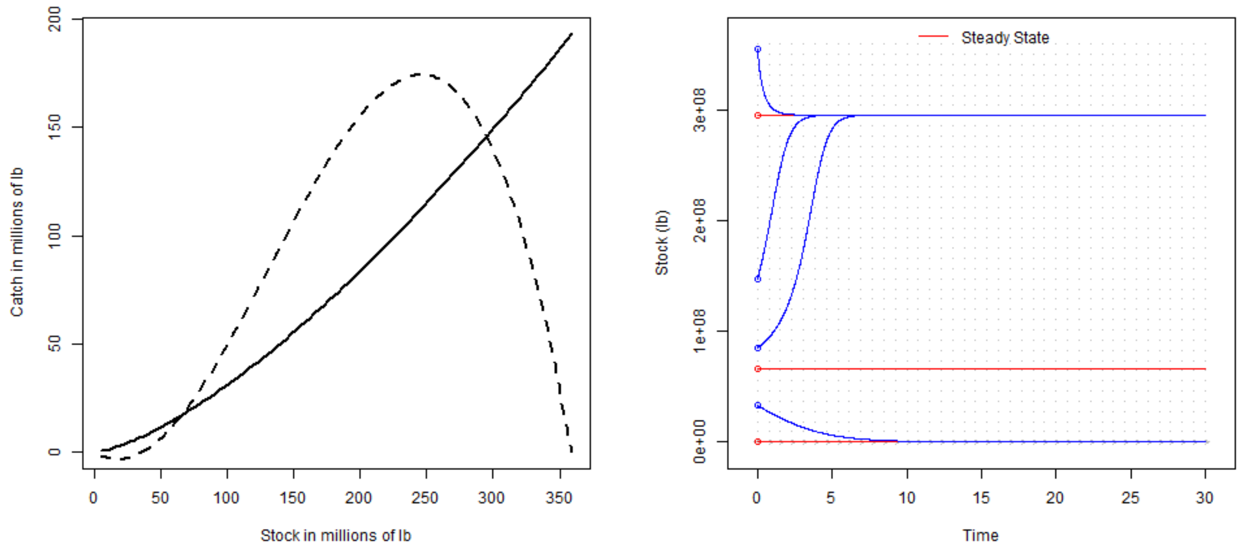
Maintaining the same harvest and profit function as before, we now adopt the following cubic growth function:

$$ds = \left(rs(t) \left(\frac{s(t)}{K_1} - 1 \right) \left(1 - \frac{s(t)}{K} \right) - h\left(x\left(s\left(t\right)\right), s\left(t\right)\right) \right) dt + \sigma s(t) dZ(t), \qquad (14)$$

where K is the same carrying capacity (3.59×10^8) as given in Equation (13) and K_1 is the minimum viable population (setting as 10 % of K). This value is set for illustrative purposes, and is not based on data. Figure (8) shows the growth function and economic program in panel (a) and the trajectories of stock, given different initial conditions, in panel (b) under deterministic conditions. Note that there are now three bioeconomic equilbria: one stable equilibrium at extinction, a stable steady state at about 295 million pounds, and an unstable steady state at about 65 million pounds. This unstable steady state – not the critical minimum viable popoulation of K_1 – marks the boundary between the basins of attraction for the two stable equilibria.

5.2. Value Function and Price Approximation

Approximation of shadow prices using basis function approximation techniques (Appendix B) for this non-convex case has proven highly unstable. Therefore, to approximate



(a) Cubic growth and economic program

(b) Phase diagram of the drift term

Figure 8: The cubic growth function and economic program and phase diagram for the cubic GOM reef fish model

the value function, we employ a straightforward but computationally intensive forward simulation method over a dense grid of 600 starting stock values to estimate the value function [18]. For each value of stock, we first generate 100,000 stochastic stock dynamics according to (14) using 800 time steps over 0.1 year intervals. We then calculate the numerical sum of the present value of profit for each trajectory, which is the Riemann sum to numerically approximate an integration with a midpoint rule. We then average over these 100,000 simulated value functions to obtain the expected value function shown in the top of Figure (9) for the deterministic case and three different levels of volatility.

To derive the shadow price functions, we fit a flexible approximation to the dense grid of value function simulations and then differentiate this approximation. Specifically, we first use the cubic spline to approximate the value function at the 500 Chebyshev polynomial nodes using the simulated values from the brute force approximation. Then, we approximate the

[18] A similar approach was suggested by (Hamilton and Ruta, 2009)

value function using an 80th order Chebyshev polynomial approximation. Exploiting the analytical differentiability of Chebyshev approximations, we use this functional approximation to approximate the shadow price function as shown in the bottom of Figure 9.[19]

The forward simulation approach is extremely robust in its ability to recover the expected intertemporal welfare function at any point in the approximation domain in the presence of strong non-linearities and stochasticity – subject only to the simulation error associated with the numerical simulation of stochastic differential equations and Monte Carlo integration, though it is not elegant or efficient. In principle, basis function approximation methods using collocation methods (i.e. where approximation nodes are equal to the number of basis coefficients to recover) exactly satisfy the HJB equation at all approximation points and utilize the flexibility of the semiparametric structure of the basis coefficients to interpolate between these coefficients – in essence solving the forward simulation at any point in the approximation domain without laborious simulations at every grid point (Fenichel, Abbott, and Yun, 2018).[20]. However, in practice we have found that basis function approximation approaches, at least using Chebyshev polynomials, perform poorly at approximating the highly non-linear shape of the intertemporal welfare function when levels, first, and second derivatives are required. Approximations are particularly poor in the highly curved (and in the deterministic case discontinuous) region around the unstable equilibrium. Furthermore, attempts to improve the quality of approximation in this region through placement of nodes or the use of additional basis functions often resulted in degeneration of the quality of the approximation in the more distant portions of the approximation domain from the unstable equilibrium.

It is possible that local, as opposed to global, approximation approaches (e.g., splines) combined with careful placement of nodes may overcome these difficulties. However, any

[19] Appendix C provides a comparison of functional approximation using Chebyshev basis coefficients to the brute force approach.

[20] Approximation using an overdetermined system (i.e. more nodes than coefficients) does not guarantee exact approximations at any point, but rather trades off approximation error at all points (Fenichel, Abbott, and Yun, 2018)

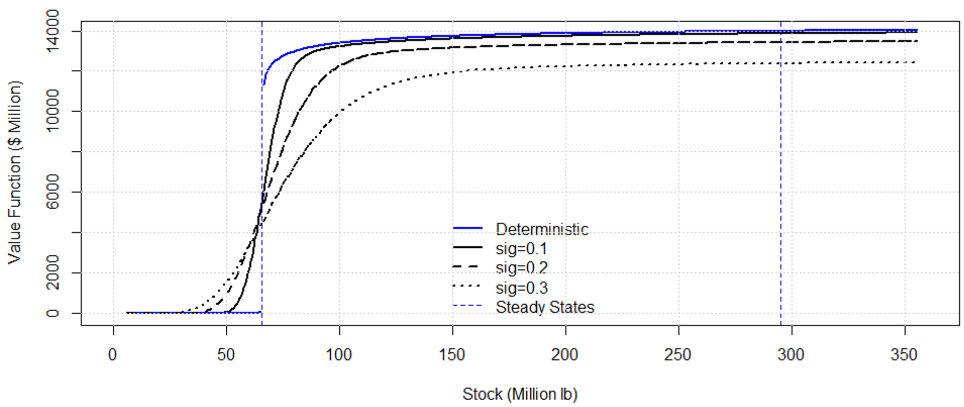
functional approximation method must ultimately grapple with the fact that the value function, lacking an analytical solution for such complex non-linear cases, is unknown – leaving open the question of the quality of the approximation in the absence of a known true value. Therefore, in order to provide robust estimates of asset values while establishing these "true" values (subject to the bounds of simulation error), we utilize the forward simulation approach in the following results. While laborious, advances in computing – particularly cost-effective parallel processing – allow the forward simulation approach to be a reasonable alternative.

5.3. Results

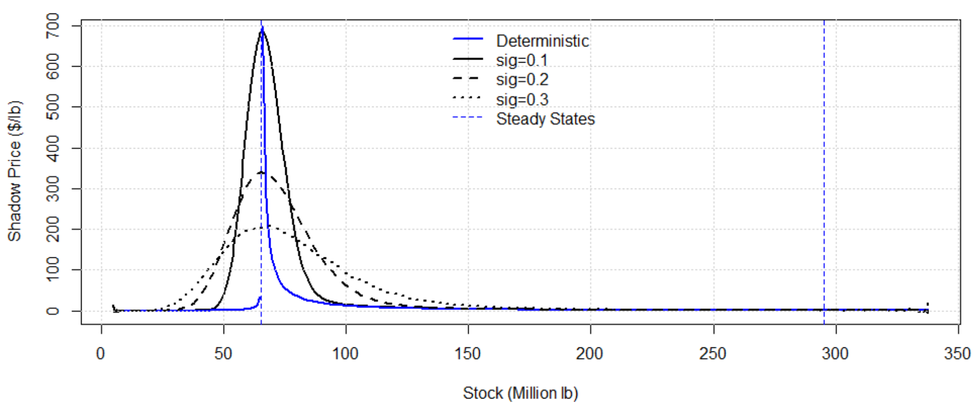
Figure 9 shows the value function under varying degrees of risk. To understand the effect of risk, it is useful to first understand the properties of the deterministic value function. First, the value function is indeterminate at the unstable equilibrium biomass, reflecting two potential trajectories in response to an infinitesimal shock. A "rightward" shock commits the system to harvests along the economic program as the stock builds to the upper stable equilibrium. The result is a concave and increasing value function in this basin of attraction, which is reflected through a uniformly positive but downward sloping price curve. Conversely, a "leftward" shock commits the system to a an inevitable drawdown of the stock to exhaustion.[21] As a result, the value function in this basin of attraction is convex and increasing in the stock. This is reflected through an increasing, albeit low, shadow price. Furthermore, the value function experiences a dramatic decline to the left of the threshold biomass, reflecting a truly catastrophic devaluation. This devaluation is also experienced at the margin, with the marginal valuation of the stock as an investment plummeting once the system is committed to the "drawdown" equilibrium.[22]

[21]Note that the economic program at very low stock levels is perhaps unrealistic, forcing fishing at low levels even when profits are negative (as in a heavily subsidized fleet) so that the value function actually becomes negative, even though this is difficult to perceive from the figure.

[22]While the context of this example is not one of optimization (the economic program is pre-determined), it is nevertheless clearly the case that if a social planner could pick a basin of attraction from the perspective of expected discounted profit maximization – in the sense of being pre-committed to the non-optimal economic program once landing in either basin – they would prefer the basin corresponding to the stable upper equilibrium to the extinction equilibrium. In other words, the value function is uniformly higher at all stock



(a) Value Function



(b) Shadow Price

Figure 9: Value function and shadow price of cubic growth GOM Reef Fish

Turning to the effects of risk, it is apparent that the addition of risk to this non-convex system has qualitative and quantitative impacts on the valuation of natural capital – in total and at the margin. Focusing on the intertemporal welfare function, the former discontinuity at the unstable equilibrium biomass is replaced by value functions that assume the shape of a logistic curve – with an inflection point at the unstable equilibrium. The steepness of the central portion of the logistic curve declines as risk increases – acting to homogenize values on either side of the threshold relative to the deterministic case. Intuitively, this effect arises because in a small neighborhood on either side of the threshold biomass, GBM stochasticity of the biomass acts symmetrically, either to throw the system into the "drawdown" basin from the "sustainable fishing" basin or vice versa. The result is a smoothing of the value function, where the extent of the smoothing increases in the level of stochasticity.

The effect of this risk-driven smoothing on the HJB equation is distinct depending upon the basin of attraction. For values of stock above the threshold value, risk leads to a negative risk adjustment term in the HJB equation (2.1), which is sensible given the concavity of the value function in this region. However, the opposite occurs below the threshold biomass, so that risk *increases* intertemporal welfare, given the concavity of the value function. Intuitively, the possibility of "good" shock to either rescue the fishery from the drawdown basin or maintain system dynamics near the threshold leads to"risk loving" in valuation in this region. Importantly, the risk premia/discounts to the HJB are at their largest in the immediate vicinity of the threshold and deteriorate as stocks move away from the threshold and the long-run value of the system is less influenced by the possibility of switching basins of attraction. This zone of influence grows in the magnitude of risk, and it persists over a wider range in the "sustainable fishing" basin due to the fact that the variance of GBM shocks increases in stock size.

The effects on shadow prices mirror those for the intertemporal welfare function. Shadow prices under risk are now roughly bell-shaped, with risk serving to homogenize marginal

levels above the unstable equilibrium relative to below it. This need not be the case in all settings.

valuations of investing in natural capital on either side of the threshold. Marginal valuations consistently peak at the threshold biomass itself and decline in either direction, with the rate of decline falling as stochasticity increases. Increasing levels of risk act to homogenize marginal valuations over a wider range of values, as reflected in the increasingly linear section of the value function in the region of the threshold. Each shadow price curve has a concave region, centered around the threshold biomass level, with two convex regions on either side. Concavity/convexity of shadow prices relate to the signs of third derivatives of the intertemporal welfare function. In other words, the qualitative effect of the second risk-related numerator term in the shadow price equation (7) varies across the domain of the stock. Within a "zone of influence" of the threshold, $V_{sss} = p_{ss} < 0$, so that risk (apart from the endogenous risk effect) tends to undermine the value of marginal investments in natural capital. In other words, the influence of the threshold induces a form of "anti-prudence" or "negative self-insurance" with respect to the valuation of the stock. This occurs because of a "lottery effect" that implies that as volatility increases current holdings of capital have less of an effect on future holdings of capital, which reduces the importance of the tradeoff between harvest today and future opportunities. The effect of the lottery in terms of one's "destiny" tends to countervail any "self-insurance" benefits of investments in natural capital. Outside of this zone of influence, however, the sign of this term reverses, yielding higher valuations reflecting a degree of prudence.[23]

The ultimate effect of all these margins on the effect of risk on marginal valuations is complex, depending strongly on the level of risk and the stock level. For a region just above the threshold biomass, adding risk unequivocally reduces the marginal valuation of capital. However, there is a biomass level at which risk has no effect on marginal valuation. Beyond this point the gap between the stochastic and deterministic marginal valuation expands and then contracts, although the shadow price under stochasticity always exceeds the determinis-

[23]The "endogenous risk" effect is likewise asymmetric on either side of the threshold, with the effect increasing shadow prices below the threshold and decreasing them above the threshold.

tic level. With respect to stock levels below the threshold level, things are more simple. Risk consistently increases the marginal value of the stock relative to the deterministic case since holding more natural capital increases the chance of escaping the lower basin of attraction.

Finally, greater levels of risk do not have monotonic effects on the marginal valuation of risk. Figure 9 clearly demonstrates that the stochastic shadow value curves cross in both basins of attraction. Therefore, near the threshold stock, marginal valuation of natural capital falls as stochasticity increases – essentially showing the growing dominance of the "lottery" effect of risk viz a viz the threshold. Yet this effect is reversed at low and high stocks beyond this zone of influence so that risk actually increases the value of investing in the stock – again, reflecting the role of prudence in the second risk term of the asset price equation (7).

While stochasticity and non-convexity may have little influence on valuation independently, the presence of such large and complex valuation effects in such a simple model suggests that it is the *interaction* between stochasticity and non-convexity that is important to understand.

6. Conclusion

Risk alters the valuation of real assets in two ways. First, investing in capital influences the strength of volatility via an endogenous risk effect. This increment in volatility is valued through the risk aversion (or risk seeking) as embodied in the curvature of the intertemporal welfare function. Second, investments may alter the curvature of the intertemporal welfare function itself, reflecting a form of endogenous risk aversion. Importantly, this risk aversion and its rate of change are not primitive features of underlying social preference, rather they are induced intertemporal preferences over capital stocks reflecting characteristics of the instantaneous welfare function, the dynamics of capital accumulation, and the economic program providing a feedback rule between capital stocks and consumption and investment behavior. The behavioral implications of these valuation effects of risk have been extensively cataloged in the macroeconomic literature – albeit primarily in the special case of dynamically optimal investment behavior, where the role of the economic program is obscured. The "dual" problem of understanding how risk influences the *valuation* of real assets, *conditional* on their management, has received far less attention.

Nowhere is the need to focus on the valuation implications of risk more important than in the case of natural capital, where absence of capital markets, market and governance failures, and sub-optimal management undermine market-derived and optimization-driven valuation approaches alike. Two unique features of natural capital – its non-linear process of accumulation (i.e. non-linear drift) and its often sub-optimal management – are likely to operate upon the risk terms in the asset pricing equation in subtle and important ways that may be distinct from those seen in the macroeconomic consumption-investment literature dominated by linear accumulation rules and dynamically optimal behavior. Through analysis of simple examples, we have shown how stochasticity influences total and marginal valuations of natural capital stocks, assessing the magnitude and direction of its effect.

Despite the rich manner in which risk *theoretically* influences the valuation of natural assets, our investigation of the single-stock, logistic model with GBM shocks found that stochasticity has only a minor impact on measures of changes in wealth for marginal and non-marginal perturbations to capital stocks along predetermined economic programs. This result is robust across optimized and non-optimized settings and for quite high (arguably unrealistic) levels of volatility. These results illustrate that the two risk terms can act in a countervailing fashion, so that the qualitative effect of risk is unclear *a priori*. It is possible that in a number of cases that these effects may approximately cancel out.[24] While our results are far from exhaustive, the weak effects of risk in a canonical resource model across a range of optimal and non-optimal control results suggests that risk may be a truly second-order concern in some important cases.

The statement that risk may be a non-issue for valuation is strongly qualified by our

[24] Indentifying these conditions is an area of important future research.

findings for our canonical example of a highly nonlinear resource system with alternative stable states. In this case we find strong and complex effects on the valuation of natural capital in the region of the transition between basins of attraction, where the zone of this influence will scale with the magnitude of the risk. Finally, the magnitude and direction of the bias from ignoring risk is likely to depend on the location of the stock within state space. These results highlight the importance of grounding natural capital valuation in the context of bioeconomic models that clarify the nature of risk's influence on valuation and the connections among natural dynamics and the revealed economic program.

The question of the effect of risk and uncertainty on natural capital valuation is quite broad in scope, and will require a research program to satisfactorily answer. Our pursuit of this question is meant as a starting place and is therefore narrow in focus, with several notable restrictions. First, we focus exclusively on risk, where stochastic processes have known probabilities. We therefore ignore concerns of ambiguity or Knightian uncertainty. Secondly, our theoretical derivations and examples are cast in continuous time and only describe stochastic processes that can be written as diffusions (i.e., Ito processes). Within this class of diffusions, we have limited our attention to GBM processes, whereas a range of other stochastic processes may offer unique insights. Furthermore, economic programs are likewise continuous in time and are functions of contemporaneous state variables. Distinct insights that might arise from consideration of discontinuous shocks (e.g., Poisson shocks) or delayed responsiveness to noisy measurements of the system. Thirdly, we focus solely on stochasticity in stock dynamics. While such "process error" is a significant concern in natural resource management, it is far from the only place where risk may be important. For example, one may be concerned about "implementation error" in the economic program, so that the intended management actions are only implemented approximately. This is of course closely related to specification error - a common concern in econometrics. Finally, while we derive asset pricing equations for multiple correlated natural capital stocks (Appendix A), our theoretical and empirical focus is solely on the single-stock case. Thus the limitations of our analysis demonstrate the many profitable avenues for elucidating the ways in which risk should alter our valuation of natural assets.

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Appendix A. Multi-stock derivation of valuation formulae

Let $\boldsymbol{s}(t) \in \mathbb{R}^S$ and $\boldsymbol{x}(\boldsymbol{s}(t)) : \mathbb{R}^S \to \mathbb{R}^X$and extend the diffusion in (1) to S distinct Ito processes

$$ds^i = \mu^i\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)dt + \sigma^i(\boldsymbol{s})dZ^i(t) \quad \text{for } i = 1, \ldots, S \tag{A.1}$$

The $dZ^i(t)$ can be correlated with a $S \times S$ correlation matrix $\boldsymbol{\rho}$ such that the covariance of the stochastic components of capital stocks i and j, which may differ from their observed covariance in-sample due to the presence of deterministic relations between the stocks in (A.1), is $\mathbb{E}_t\left[\sigma^i(\boldsymbol{s})dZ^i(t)\sigma^j(\boldsymbol{s})dZ^j(t)\right] = \sigma^i(\boldsymbol{s})\sigma^j(\boldsymbol{s})\mathbb{E}_t\left[dZ^i(t)dZ^j(t)\right] = \sigma^i(\boldsymbol{s})\sigma^j(\boldsymbol{s})\rho^{ij}dt$. If $i = j$, then the expression simplifies to $\sigma^i(\boldsymbol{s})^2dt$.

While the decomposition of the noise into a correlation matrix and standard deviations is intuitive and useful for model parameterization, we work directly with the covariance matrix to conserve on notation. Let $\Omega(\boldsymbol{s})$ be a $S \times S$ covariance matrix of the noise terms such that $\text{Cov}(ds^i, ds^j) = \Omega^{ij}(\boldsymbol{s})\, dt$. A Cholesky decomposition of the covariance matrix yields $\Omega(\boldsymbol{s}) = \omega(\boldsymbol{s})\omega(\boldsymbol{s})'$.[25]

Redefine the instantaneous return functions and intertemporal welfare functions in the multi-stock case as $W\left(\boldsymbol{s}\left(t\right), \boldsymbol{x}\left(\boldsymbol{s}\left(t\right)\right)\right)$ and $V(\boldsymbol{s}(t))$. Once again, we know that $\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt}$. Applying Ito's Lemma (Dixit and Pindyck, 1994) yields:

$$dV(\boldsymbol{s}) = \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s})V_{s^j s^k}\right]dt + \sum_{j=1}^{S} \sigma^j(\boldsymbol{s})V_{s^j}dZ^j$$

Finding the expected value and dividing through by dt:

$$\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt} = \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s})V_{s^j s^k}\right] \tag{A.2}$$

[25] This approach generalizes (A.1) slightly by technically allowing for the *correlation* matrix - not just the standard deviations - to vary in the stock vector.

Setting (A.2) equal to the multidimensional generalization of (3) yields the HJB equation.

$$\delta V(\boldsymbol{s}) = W\left(\boldsymbol{s}\left(t\right), \boldsymbol{x}\left(\boldsymbol{s}\left(t\right)\right)\right) + \left[\sum_{j=1}^{S} \mu^{j}\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] \tag{A.3}$$

Partial differentiation of (A.3) yields the following expression for the shadow price of s^i

$$p^i(\boldsymbol{s}) = \frac{W_{s^i} + \left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^{S} \frac{\partial p^j}{\partial s^i}\mu^j\right) + \sum_{j\neq i}^{S} p^j \mu^j_{s^i} + \frac{1}{2}\sum_{j}^{S}\sum_{k}^{S}\left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)}{\delta - \mu^i_{s^i}}$$

Factoring the final numerator term yields the final asset pricing equation.

$$\begin{aligned} p^i(\boldsymbol{s}) = \Bigg[& W_{s^i} + \left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^{S}\frac{\partial p^j}{\partial s^i}\mu^j\right) + \sum_{j\neq i}^{S} p^j\mu^j_{s^i} + \frac{1}{2}\sum_{j=1}^{S}\left(\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j} + \sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}\right) \\ & + \frac{1}{2}\sum_{j=1}^{S}\sum_{k\neq j}^{S}\left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)\Bigg] \Bigg/ \left(\delta - \mu^i_{s^i}\right) \end{aligned} \tag{A.4}$$

The first numerator term in (A.4) has the same interpretation as in the single-asset case. The next two terms in the numerator are present in the deterministic multi-asset case (Yun et al., 2017) and are forms of "capital gains." The second numerator term $\left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^{S}\frac{\partial p^j}{\partial s^i}\mu^j\right)$ reflects the effects of investment in s^i on the shadow price of stock i due to its prices of all assets in the portfolio (i.e. "price effects"). The third numerator term $\sum_{j\neq i}^{S} p^j\mu^j_{s^i}$ captures the deterministic effects of investment in stock i on the physical growth rates of all other stocks ("cross-stock effects"), which can stem from system ecology or production interactions within the economic program.

The additional numerator terms in (A.4) only exist in the stochastic case. The third term $\frac{1}{2}\sum_{j=1}^{S}\left(\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j} + \sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}\right)$ operates solely through the individual variances of each asset and captures the "risk sensitivity" effect of an investment in asset i on the variance of each asset, $\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j}$. This part of the term reflects how substitution and complementarity relationships can provide "self-protection" through "portfolio diversification," which is the endogenous risk concept. Importantly, the $\sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}$ term represents prudence and accounts for the fact that investments in i also affects the *sensitivity* to risk for all S assets, $\sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}$, even if the variance for these other assets remains unchanged by the investment. This means that this term influences the consequences of stochastic events, and can be thought of as a self-insurance term. Together, these terms mirror the numerator terms, $\sigma(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}$, in (7).

The final term in the numerator of (A.4), $\frac{1}{2}\sum_{j=1}^{S}\sum_{k\neq j}^{S}\left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)$, reflects the risk-related effects of investing in asset i that are mediated through the *covariances* of assets in the portfolio. This term is zero in the case that natural capital stocks are uncorrelated regardless of the vector of capital stocks. $\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k}$ is the effect of an investment in i on the covariances between other assets j, k as valued through the first cross-partial between these assets (i.e. the 2nd cross-partial of the intertemporal welfare function). If the covariances between asset stocks are invariant to capital stocks then this term is zero. $\Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}$ reflects the fact that investing in i may itself affect the curvature of the intertemporal welfare function in the direction of k and i (i.e. $\frac{\partial^2 p^j}{\partial s^k \partial s^i} = \frac{\partial}{\partial s^i}V_{s^j s^k}$). If the effect of increasing asset i is to increase the concavity in the direction of increases in j and k ($\frac{\partial^2 p^j}{\partial s^k \partial s^i} < 0$) then the existence of positive correlation between the latter two assets results in a compensating reduction in the asset price. This creates addition "self insurance" opportunities from portfolio diversification.

Some insight on the numerator terms involving covariances can be gleaned by realizing that the covariance between innovations in s^j (the residual of changes in s^j after the deterministic drift $\mu^j(\boldsymbol{s}, \boldsymbol{x}(\boldsymbol{s}))$ is differenced away) and innovations in s^k can be viewed as their rescaled relationship in expectation. Specifically, if the conditional expectation of s^j and s^k is linear[26] $\mathbb{E}[ds^j|ds^k] = \beta ds^k$, then it is well known that $\beta = \frac{\Omega^{jk}}{\sigma^{2k}}$. In other words, the covariance terms in (A.4) reflect the expected marginal effect of ds^k on ds^j such that the risk terms in the multivariate asset case account for systematic (linear) cross-effects be-

[26] Linearity of the conditional mean follows directly from the joint normality assumption for Ito processes.

tween perturbations in stocks in a way that is analogous to how the previous cross-terms in the numerator account for capital gains through deterministic relationships via price and cross-stock effects.

Finally, it is noteworthy that the effects of stochasticity disappear from (A.4) when two conditions hold: 1) when all second moments are constant regardless of the stock levels, and 2) the intertemporal welfare function, V, is quadratic such that investments have no effect on its curvature. However, since the intertemporal welfare function inherits the properties of the instantaneous benefits function, the economic program, and biophysical dynamics in a complex manner, the latter property is difficult to verify *ex ante*.

The numerical approximation of the shadow price function is carried out using "value function approximation" and is detailed in **??**. As detailed in Fenichel, Abbott, and Yun (2018) for the deterministic case and employed in Yun et al. (2017), this approach uses a Chebyshev polynomial basis to approximate the intertemporal welfare function using the HJB equation. We then differentiate the HJB equation to obtain estimates of the shadow prices.

Appendix B. Numerical approximation

Fenichel, Abbott, and Yun (2018) and Yun et al. (2017) describe how the HJB equation can be combined with functional approximation approaches frequently used in numerical dynamic programming to approximate the entire shadow price *function* over a closed domain of capital stocks. For the deterministic, multi-asset case they advocate approximating $V(\boldsymbol{s}(t))$ using the HJB equation (analogous to (A.3)), replacing $V(\boldsymbol{s}(t))$ on the LHS of the equation with a weighted sum of the tensor product of Chebyshev basis functions in the stock vector $\boldsymbol{s}(t)$ and replacing the partial derivatives of the value function on the RHS with the partial derivatives of this approximation. The coefficients that determine the weightings on the basis functions can be solved analytically and are chosen (in a system with as many approximation points as coefficients) to make the LHS and RHS of the approximated HJB equation hold with equality.[27]

This value (intertemporal welfare) function approximation technique can be adapted with relatively minor changes to the stochastic diffusion case. First, define the bounded approximation interval for each state variable. Then choose M evaluation points within this interval for each of the S capital stocks and then calculate $W\left(\boldsymbol{s}\left(t\right), x\left(\boldsymbol{s}\left(t\right)\right)\right)$, $\mu\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)$ and $\Omega(\boldsymbol{s})$ at each point.[28] The univariate node coordinates are then permuted to yield M^S grid points. We define ϕ^i as the $M \times (q^i + 1)$ basis matrix of q^ith degree for state variable i. This is a matrix of $q^i + 1$ basis functions - Chebyshev polynomials of ascending degree in our case - evaluated at the M evaluation points. To approximate over the bounded domain in $\mathbb{R}^S$ we find the tensor product across all dimensions (i.e. allow for full interactions across the univariate basis functions) to form an $M^S \times \prod_{i=1}^{S}\left(q^i + 1\right)$ basis matrix: $\boldsymbol{\Phi}\left(\boldsymbol{S}\right) = \phi^N \otimes \phi^{N-1} \otimes \ldots \otimes \phi^1$ where $\boldsymbol{S}$ is the $M^S \times S$ matrix of evaluation points (i.e. all grid nodes of M evaluation points for all S state variables). We can now define our approximation to the intertemporal welfare function $V\left(\boldsymbol{S}^m\right) \approx \boldsymbol{\Phi}^m\left(\boldsymbol{S}\right)\boldsymbol{\beta}$ where m indexes the M^S distinct capital stock vectors (i.e. the individual evaluation points in the S-dimensional grid) and $\boldsymbol{S}^m$ is the mth row of $\boldsymbol{S}$. $\boldsymbol{\Phi}^m\left(\boldsymbol{S}\right)$ is the mth row of $\boldsymbol{\Phi}\left(\boldsymbol{S}\right)$, and $\boldsymbol{\beta}$ is a $\prod_{i=1}^{S}\left(q^i + 1\right) \times 1$ vector of unknown approximation coefficients. Using the fact that $\frac{\partial V(\boldsymbol{S}^m)}{\partial s^i} \approx \left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^i}\right)\boldsymbol{\beta}$ and $\frac{\partial^2 V(\boldsymbol{S}^m)}{\partial s^i \partial s^j} \approx \left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^i \partial s^j}\right)\boldsymbol{\beta}$ we can replace the HJB equation in (A.3) with the following

[27] In some cases it may be desirable to utilize more approximation nodes than the number of coefficients - an over-determined system. In this case, the coefficients can be chosen to minimize the sum of squared deviations between the LHS and RHS of the approximation. The analytical expression for this solution is analogous to ordinary least squares (Fenichel, Abbott, and Yun, 2018).

[28] In many cases the evaluation nodes are found by finding the M roots of a unidimensional Chebyshev polynomial on the bounded approximation range for each state variable. However, care must be taken so that the nodes are laid out in a way that the system dynamics do not leave the approximating domain in expectation.

approximation:

$$\delta \boldsymbol{\Phi}^m(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S}^m) + \left[\sum_{j=1}^{S} diag(\mu^j(\boldsymbol{S}^m))\left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j}\right)\boldsymbol{\beta} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} diag(\Omega^{jk}(\boldsymbol{S}^m))\left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j \partial s^k}\right)\boldsymbol{\beta}\right] \tag{B.1}$$

Collecting terms involving β yields:

$$\left[\delta \boldsymbol{\Phi}^m(\boldsymbol{S}) - \sum_{j=1}^{S} diag(\mu^j(\boldsymbol{S}^m))\left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j}\right) - \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} diag(\Omega^{jk}(\boldsymbol{S}^m))\left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j \partial s^k}\right)\right]\boldsymbol{\beta}$$

$$= \boldsymbol{\Psi}^m(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S}^m)$$

Stacking these M^S vector equations results in the equation $\boldsymbol{\Psi}(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S})$. If $M^S = \prod_{i=1}^{S}(q^i + 1)$ (i.e. the number of approximation points equals the number of unknown approximation coefficients) then the approximation coefficients can be calculated in a straight foward way through matrix inversion. Alternatively, if $M^S > \prod_{i=1}^{S}(q^i + 1)$ then the $\boldsymbol{\beta}$ can be found using least squares.

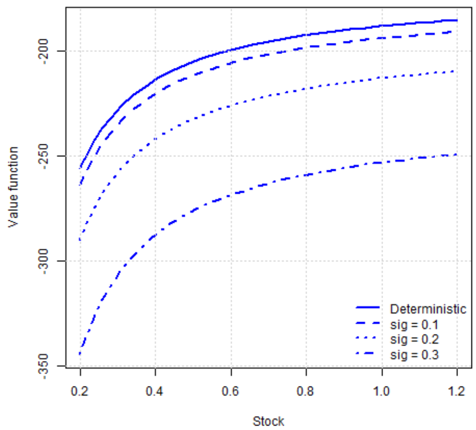
$$\boldsymbol{\beta} = \left(\boldsymbol{\Psi}(\boldsymbol{S})'\boldsymbol{\Psi}(\boldsymbol{S})\right)^{-1}\boldsymbol{\Psi}(\boldsymbol{S})'W(\boldsymbol{S}) \tag{B.2}$$

After obtaining the approximation $\boldsymbol{\Phi}(\boldsymbol{S})$ it is straightforward to find the shadow values of any given capital stock by taking its partial derivative.

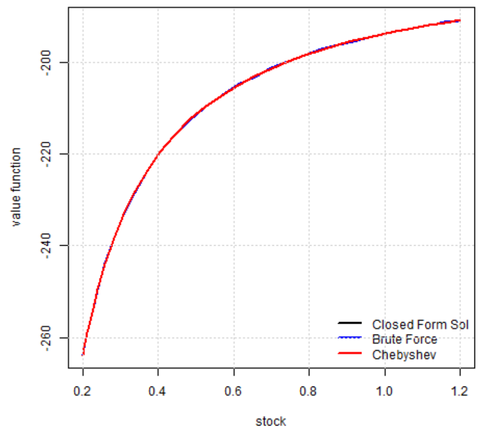
Fenichel, Abbott, and Yun (2018) discuss the importance of determining the domain of approximation. They show that in multi-dimensional systems the system dynamics to can lead outside the approximation domain, which hinders the ability to recover shadow prices. They argue that it is important to make sure the approximation domain is sufficient to include dynamic from any stock size for which a shadow price is desired. In the single stock deterministic case this is never an issue so long as the system has attractors that are within the approximation domain. However, this property does not extend to stochastic dynamics. This is because a shock at the edge of the approximation domain could lead the system outside the approximation domain for a non-trivial period of time. Therefore, extra attention is likely needed to enlarge the approximation domain when system dynamics are stochastic.

Appendix C. Numerical Performance Comparison: Chebyshev vs. Brute Force

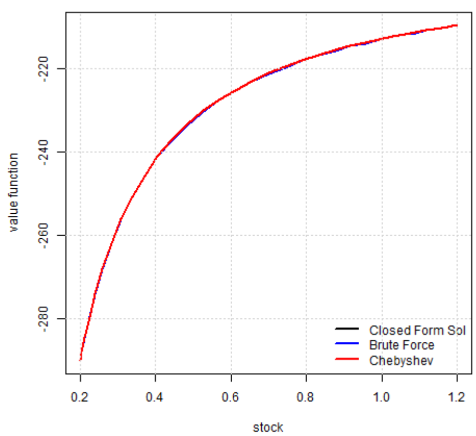
The Pindyck example presented in Section 3 is useful since it provides a known closed-form solution for the intertemporal welfare function. We use it to compare the accuracy of the numerical approximation of Chebyshev polynomial approximation and the "brute force" forward simulation approach. For this comparison, 35 Chebyshev polynomial nodes for 35th order (exactly identified) are applied for Chebyshev polynomial approximation. In the brute forth approximation, 0.01 time interval for the final time 200 are replicated for 20000 stochastic simulations. The graphical comparison across the closed form solution, Chebyshev polynomial approximations, and brute force forward simulation are shown in Figure (C.10). Panel (a) of Figure (C.10) is the closed form solution of the value function for the deterministic and stochastic models. The other panels in Figure (C.10) are a graphical comparison of the closed form solution (black), Chebyshev polynomial approximation (red), and brute force forward simulation (blue) across different volatility levels.



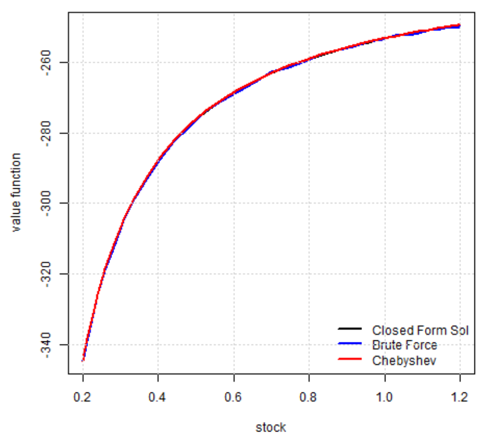
(a) Closed form solution



(b) $\sigma = 0.1$



(c) $\sigma = 0.2$



(d) $\sigma = 0.3$

Figure C.10: Comparisons of numerical approximations of Pindyck model: Closed form solution vs. Chebyshev polynomial approximation and brute force forward simulation.

Table C.2: Comparison of numerical performances: Chebyshev polynomial and brute force approximation results are compared with the closed form solution.

	Chebyshev		Burte Force	
	RMSE	Running Time	RMSE	Running Time
$\sigma = 0.1$	4.19e-07	0.11 sec.	0.0096	121.40 min.
$\sigma = 0.2$	1.95e-05	0.84 sec.	0.0650	126.54 min.
$\sigma = 0.3$	0.0041	0.10 sec.	0.1336	127.11 min.

Table C.2 presents the numerical performances of Chebyshev polynomial and brute force approximation. It is obvious that Chebyshev polynomial approximation dominates the error size and running time for all volatility cases. If Chebyshev polynomial approximation works, its approximation performances are expected to be better than that brute force approximation provides. However, Chebyshev polynomial approximation often crashes its performance (e.g., our cubic growth example) or produces overfitting issues (e.g., overly wavy curves). Since the true value function is unknown in many general cases, Chebyshev polynomial approximation is not always applicable or the best. This is generally true for the most of global approximation methods not limited to Chebyshev polynomial approximation. Numerical performances of the brute force approximation is really variable. Even though its expensive computational burden, the brute force approximation has its own value. First of all, brute force approximation is not likely to fail for producing its results. This means that brute force approximation is mostly applicable for where the global approximation methods fails. Also, the brute force approximation can provide the best guess about how the true value function is shaped. With the abundant computational resources, the approximation error size could be reduced extremely. With $\sigma = 0.1$ simulation, the RMSE is 0.1330, 0.0096, 1.81e-05 and for 10000, 20000, and 30000 stochastic simulations. Using the parallel computing resources, the approximation error size and computation time could be reduced in a favorable level. As an independent approximator or first snapshot provider of value function, in general, the brute force approach compares favorably to the Chebyshev approach (or any other global approximation approaches). In the conference version of this paper we will have a complete numerical comparison of these methods.